MATH 6338 (C. Heil) HOMEWORK #1 Due date: May 30, 2008

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let X be a normed space. Prove that X is a Banach space if and only if every absolutely convergent series in X converges in X.

Definition 1. We say that a function $f\colon \mathbb{R} \to \mathbb{C}$ is *Hölder continuous* with exponent $\alpha > 0$ if there exists a constant $K > 0$ such that

$$\forall\, x, y \in \mathbb{R}, \quad |f(x) - f(y)| \;\le\; K\,|x-y|^{\alpha}.$$

2. Given $0 < \alpha < 1$, define

$$C^{\alpha}(\mathbb{R}) \;=\; \bigl\{f \in C(\mathbb{R}) : f \text{ is Hölder continuous with exponent } \alpha\bigr\}.$$

Show that

$$\|f\|_{C^{\alpha}} \;=\; |f(0)| \;+\; \sup_{x \neq y} \frac{f(x)-f(y)}{|x-y|^{\alpha}}$$

is a norm on $C^{\alpha}(\mathbb{R})$, and that $C^{\alpha}(\mathbb{R})$ is complete with respect to this norm.

3. Show that if $|E| < \infty$ and $0 < p \le q \le \infty$, then $L^q(E) \subseteq L^p(E)$, with

$$\|f\|_p \;\le\; \|f\|_q\,|E|^{\frac{1}{p}-\frac{1}{q}}, \qquad f \in L^p(\mathbb{R}).$$

In contrast, show that $L^p(\mathbb{R})$ is not contained in $L^q(\mathbb{R})$ for any $p \neq q$, regardless of which is larger.

4. Suppose that $E \subseteq \mathbb{R}^d$ is Lebesgue measurable and satisfies $|E| < \infty$. Show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

5. Show that ℓ^p is separable when $p < \infty$.