MATH 6338 (C. Heil) HOMEWORK #2 Due date: June 11, 2008

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let X, Y, Z be normed linear spaces.

(a) Prove that if $L\colon X \to Y$ is linear, then $\|L\| = \inf\{K > 0 : \forall\, x \in X,\ \|Lx\| \le K\|x\|\}$.

(b) Prove that the operator norm is a norm on $\mathcal{B}(X, Y)$.

(c) Prove that the operator norm is submultiplicative, i.e., if $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, Z)$, then $\|BA\| \le \|A\|\,\|B\|$.

2. Let H be a Hilbert space. For each $h \in H$, define $\mu_h\colon H \to \mathbb{C}$ by

$$\mu_h(f) \;=\; \langle f, h\rangle, \quad f \in H.$$

(a) Show that $\mu_h \in H^*$ for each $h \in H$, and that $\|\mu_h\| = \|h\|$.

(b) Define $T\colon H \to H^*$ by $T(h) = \mu_h$. Prove that T is an injective, antilinear, isometric map of H into H^*. In particular,

$$\mu_{\alpha g+\beta h} \;=\; \bar{\alpha}\mu_g + \bar{\beta}\mu_h.$$

We will show in class that T is also surjective (this is the *Riesz Representation Theorem*).

Definition 1. Let X be a Banach space.

(a) A sequence $\{f_n\}_{n\in\mathbb{N}}$ in X is *ω-independent* if there do not exist scalars c_n, not all zero, such that $\sum_{n=1}^{\infty} c_n f_n = 0$, where the series converges in the norm of X.

(b) A sequence $\{f_n\}_{n\in\mathbb{N}}$ is a *Schauder basis* for X if for each $f \in X$ there exist unique scalars $c_n(f)$ such that $f = \sum_{n=0}^{\infty} c_n f_n$, where the series converges in the norm of X.

3. Let X be a Banach space.

(a) Show that every Schauder basis for X is complete and ω-independent.

(b) Let α, $\beta \in \mathbb{C}$ be fixed nonzero scalars such that $\left|\frac{\alpha}{\beta}\right| > 1$. Define

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \dots),\\ x_1 &= (\alpha, \beta, 0, 0, \dots),\\ x_2 &= (0, \alpha, \beta, 0, \dots), \end{aligned}$$

etc. Prove that $\{x_k\}_{k\ge 0}$ is complete and finitely linearly independent in ℓ^2, but is not ω-independent and therefore is not a Schauder basis for ℓ^2.

4. Define $e_n(x) = e^{2\pi inx}$. The sequence $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1]$ (you can take that as given). If $f \in L^2[0,1]$ then $\widehat{f}(n) = \langle f, e_n\rangle$ are the *Fourier coefficients* of f. Compute the Fourier coefficients of the function $f = \chi_{[0,1/2)} - \chi_{[1/2,1)}$, and observe that $\widehat{f} \in \ell^2(\mathbb{Z}) \setminus \ell^1(\mathbb{Z})$. Apply the Plancherel equality and derive a formula for π.

5. The *convolution* of two functions f and g is the function $f * g$ given by

$$(f * g)(x) \;=\; \int f(y)\, g(x-y)\, dy,$$

whenever this makes sense. Use Hölder's Inequality to give a direct proof of *Young's Inequality*: If $1 \le p \le \infty$, then

$$\forall\, f \in L^p(\mathbb{R}), \quad \forall\, g \in L^1(\mathbb{R}), \quad \|f * g\|_p \;\le\; \|f\|_p\, \|g\|_1.$$

You may assume without proof that these hypotheses imply that $f * g$ exists and is measurable.

Hint: Write

$$|(f * g)(x)| \;\le\; \int \Bigl(\bigl|f(y)\bigr|\, \bigl|g(x-y)\bigr|^{1/p}\Bigr)\, \bigl|g(x-y)\bigr|^{1/p'}\, dy,$$

and apply Hölder's Inequality.

6. The *Fourier transform* of a function $f \in L^1(\mathbb{R})$ is

$$\widehat{f}(\xi) \;=\; \int f(x)\, e^{-2\pi i\xi x}, \qquad \xi \in \mathbb{R}.$$

(a) Define $\mathcal{F}(f) = \widehat{f}$. Prove that $\mathcal{F}$ is a bounded mapping of $L^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, and that $\|\mathcal{F}\| \le 1$.

(b) Show that if $f,\, g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ and $(f * g)^\wedge(\xi) = \widehat{f}(\xi)\, \widehat{g}(\xi)$.