MATH 6338 (C. Heil) HOMEWORK #3 Due date: June 30, 2008

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let M be a closed subspace of a normed linear space X.

(a) Prove that if X is separable, then X/M is separable.

(b) Prove that if X/M and M are both separable, then X is separable.
Hint: Let $\{f_n+M\}_{n\in\mathbb{N}}$ be a countable dense subset of X/M, and let $\{g_n\}_{n\in\mathbb{N}}$ be a countable dense subset of M. Then $S = \{f_m + g_n\}_{m,n\in\mathbb{N}}$ is a countable subset of X.

(c) Give an example of X, M such that X/M is separable, but X is not separable.

2. Let X be a Banach space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in X. Show that the following two statements are equivalent. Hint: Hahn–Banach.

(a) $\{x_n\}_{n\in\mathbb{N}}$ is *minimal*, i.e., no x_m lies in the closed span of the other vectors in the sequence:

$$\forall\, m \in \mathbb{N}, \quad x_m \notin \overline{\mathrm{span}}\{x_n\}_{n\neq m}.$$

(b) There exists a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in X^* that is *biorthogonal* to $\{x_n\}_{n\in\mathbb{N}}$, i.e.,

$$\langle x_n, \mu_m\rangle \;=\; \delta_{mn} \;=\; \begin{cases} 1, & m = n,\\ 0, & m \neq n. \end{cases}$$

3. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a Banach space X. Consider the following two statements. Prove that statement (a) implies statement (b). (In fact, the converse is also true, but you don't need to prove it.)
Hint: Hahn–Banach.

(a) $\sum_{n=1}^{\infty} |\langle x_n, \mu\rangle|$ converges uniformly with respect to the unit sphere in X^*, i.e.,

$$\lim_{N\to\infty} \left(\sup_{\|\mu\|=1} \sum_{n=N}^{\infty} |\langle x_n, \mu\rangle| \right) \;=\; 0.$$

(b) $\sum_{n=1}^{\infty} c_n x_n$ converges for every sequence $(c_n)_{n\in\mathbb{N}} \in \ell^\infty$.

4. Let M be a closed subspace of a Banach space X. Let $\rho_X\colon X \to X^{**}$ and $\rho_M\colon M \to M^{**}$ be the natural maps. Let $i\colon M \to X$ be the inclusion map (i.e., $i(x) = x$ for $x \in M$). Prove that there exists an isometry $\phi\colon M^{**} \to X^{**}$ such that $\rho_X \circ i = \phi \circ \rho_M$.