MATH 6338 (C. Heil) HOMEWORK #4 Due date: July 11, 2008

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let X, Y be Banach spaces, and let $A \in \mathcal{B}(X,Y)$ be fixed. Show that there exists a unique operator $A^* \in \mathcal{B}(Y^*,X^*)$ that satisfies

$$\forall\, x \in X, \quad \forall\, \mu \in Y^*, \quad \langle Ax, \mu\rangle \;=\; \langle x, A^*\mu\rangle. \tag{1}$$

Show further that

$$\|A^*\| \;=\; \|A\|.$$

2. Show that the Baire Category Theorem is equivalent to the following statement: If X is a complete metric space and $U_n \subseteq X$ is dense and open for $n \in \mathbb{N}$, then $\cap U_n$ is dense in X.

3. Show that if X is an infinite-dimensional Banach space, then any Hamel basis for X must be uncountable.

Remark: A Hamel basis is an ordinary vector space basis, i.e., its finite linear span is X and every finite subset is linearly independent.

Definition. Let X be a Banach space, and let f_n, f be vectors in X. Then we say that f_n *converges weakly to* f, denoted $f_n \xrightarrow{\text{w}} f$, if

$$\forall\, \mu \in X^*, \quad \lim_{n\to\infty} \langle f_n, \mu\rangle \;=\; \langle f, \mu\rangle.$$

Definition/Theorem. $M_b(\mathbb{R})$ is the space of all complex Borel measures on $\mathbb{R}$. This is a Banach space with respect to the norm $\|\nu\| = |\nu|(\mathbb{R})$, where $|\nu|$ is the total variation measure of ν.

Riesz Representation Theorem. $C_0(\mathbb{R})^* \cong M_b(\mathbb{R})$. Specifically each complex measure $\nu \in M_b(\mathbb{R})$ defines a bounded linear functional on $C_0(\mathbb{R})$ via

$$\langle f, \nu\rangle \;=\; \int f(x)\, d\bar{\nu}(x) \;=\; \overline{\int \overline{f(x)}\, d\nu(x)}, \qquad f \in C_0(\mathbb{R}),$$

and every bounded linear functional on $C_0(\mathbb{R})$ has this form for some measure $\nu \in M_b(\mathbb{R})$.

4. Let $f_n,\ f \in C_0(\mathbb{R})$. Show that $f_n \xrightarrow{\text{w}} f$ (weak convergence) in $C_0(\mathbb{R})$ if and only if $f_n(x) \to f(x)$ pointwise for each x and $\sup \|f_n\|_\infty < \infty$.