See also Section 5.1 in Folland.

A

Metrics, Norms, Inner Products, ~~Topology~~ Banach Spaces

These appendices collect the background material needed for the main part of the volume. They are intended as a substantial review but not a complete development. Proofs and exercises are only given for selected material, where they illuminate the definitions or the main results and are reasonably self-contained. Longer or more difficult proofs are often omitted, and major theorems may be stated without proof.

A.1 Notational Conventions

We first mention a few notational conventions that are used throughout this volume.

Everything is analogous for vector spaces over the real line $\mathbb{R}$.

Unless otherwise specified, all vector spaces in this volume are taken over the complex field $\mathbb{C}$. In particular, functions whose domain is the real line $\mathbb{R}$ are generally allowed to take values in the complex plane $\mathbb{C}$.

Integrals with unspecified limits are taken over either the real line or $\mathbb{R}^d$, according to context. Thus, if $f\colon \mathbb{R} \to \mathbb{C}$, then we take

$$\int f(x)\,dx \;=\; \int_{-\infty}^{\infty} f(x)\,dx.$$

The extended real line is $\mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$. We use the conventions that $1/0 = \infty$, $1/\infty = 0$, $0 \cdot \infty = 0$.

If $1 \le p \le \infty$ is given, then its *dual index* or *dual exponent* is the extended real number p' that satisfies

$$\frac{1}{p} + \frac{1}{p'} \;=\; 1.$$

Explicitly,

$$p' \;=\; \frac{p}{p-1}.$$

The dual index lies in the range $1 \le p' \le \infty$, and we have $1' = \infty$, $2' = 2$, and $\infty' = 1$.

A.2 Metrics and Convergence

A metric determines a notion of distance between points in a set.

Definition A.1 (Metric Space). Let X be a set. A *metric* on X is a function $\mathrm{d}\colon X \times X \to \mathbb{R}$ such that:

(a) $\mathrm{d}(f,g) \geq 0$ for all $f,\, g \in X$,
(b) $\mathrm{d}(f,g) = 0$ if and only if $f = g$,
(c) $\mathrm{d}(f,g) = \mathrm{d}(g,f)$ for all $f,\, g \in X$,
(d) Triangle Inequality: for all $f,\, g,\, h \in X$ we have

$$\mathrm{d}(f,h) \;\leq\; \mathrm{d}(f,g) + \mathrm{d}(g,h).$$

In this case, X is a called a *metric space*. The value $\mathrm{d}(f,g)$ is the *distance* from f to g.

If we need to explicitly identify the metric we write "let X be a metric space with metric d" or "let (X,d) be a metric space." Often, the metric is clear from context.

A metric space need not be a vector space, although this will be true of most of the metric spaces encountered in this volume.

Once we have a notion of distance, we have a corresponding notion of convergence.

Definition A.2 (Convergent and Cauchy Sequences). Let X be a metric space with metric d, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.

(a) We say that $\{f_n\}_{n\in\mathbb{N}}$ *converges* to $f \in X$ if $\lim\limits_{n\to\infty} \mathrm{d}(f_n, f) = 0$, i.e., if

$$\forall \varepsilon > 0, \quad \exists N > 0, \quad \forall n \geq N, \quad \mathrm{d}(f_n, f) < \varepsilon.$$

In this case, we write $\lim\limits_{n\to\infty} f_n = f$ or $f_n \to f$.

(b) We say that $\{f_n\}_{n\in\mathbb{N}}$ is *Cauchy* if

$$\forall \varepsilon > 0, \quad \exists N > 0, \quad \forall m,n \geq N, \quad \mathrm{d}(f_m, f_n) < \varepsilon.$$

Exercise A.3. Let X be a metric space.

(a) Every convergent sequence in X is Cauchy.
(b) The limit of a convergent sequence is unique.

In general, however, a Cauchy sequence need not be convergent (cf. Exercises A.60–A.62).

Definition A.4 (Complete Metric Space). If every Cauchy sequence in a metric space X has the property that it converges to an element of X, then X is said to be *complete*.

Beware that the term "complete" is heavily overused and has a number of distinct mathematical meanings (see especially the terminology for *complete sequences* introduced in Definition A.72).

We will use the following notation.

Notation A.5. Let X be a metric space. Given $f \in X$ and $r > 0$, the *open ball* in X of radius r centered at f is

$$B_r(f) = \{g \in X : \mathrm{d}(f,g) < r\}. \tag{A.1}$$

The terminology "open" is related to the topology induced by the metric, see Section A.6.

A.3 Norms and Seminorms

A norm provides a notion of the length of a vector in a vector space.

Definition A.6 (Seminorms and Norms). Let X be a vector space over the field $\mathbb{C}$ of complex scalars. A *seminorm* on X is a function $\|\cdot\|\colon X \to \mathbb{R}$ such that for all $f, g \in X$ and all scalars $c \in \mathbb{C}$ we have:

(a) $\|f\| \geq 0$,

(b) $\|cf\| = |c|\,\|f\|$, and

(c) Triangle Inequality: $\|f + g\| \leq \|f\| + \|g\|$.

A seminorm is a *norm* if we also have:

(d) $\|f\| = 0$ if and only if $f = 0$.

A vector space X together with a norm $\|\cdot\|$ is called a *normed linear space* or simply a *normed space*.

Note that if S is a subspace of a normed space X, then S is itself a normed space with respect to the norm on X (restricted to S).

The following exercise shows that all normed spaces are metric spaces. In particular, the notions of convergent and Cauchy sequences apply in any normed space.

Exercise A.7. If X is a normed space, then

$$\mathrm{d}(f,g) = \|f - g\|$$

defines a metric on X, called the *induced metric*.

The Schwartz space (see Sections 1.9 and ??) is an example of a metric space whose metric is not induced from any norm; another is ℓ^p with $0 < p < 1$ (see Section A.4).

Exercise A.8. Show that if X is a normed linear space, then the following statements hold.

(a) Reverse Triangle Inequality: $\big|\|f\| - \|g\|\big| \le \|f - g\|$.
(b) Continuity of the norm: $f_n \to f \implies \|f_n\| \to \|f\|$.
(c) Continuity of vector addition: $f_n \to f$ and $g_n \to g \implies f_n + g_n \to f + g$.
(d) Continuity of scalar multiplication: $f_n \to f$ and $\alpha_n \to \alpha \implies \alpha_n f_n \to \alpha f$.
(e) Boundedness of convergent sequences: if $\{f_n\}_{n\in\mathbb{N}}$ is convergent then $\sup \|f_n\| < \infty$.
(f) Boundedness of Cauchy sequences: if $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy then $\sup \|f_n\| < \infty$.

Definition A.9 (Banach Space). A normed linear space X is called a *Banach space* if it is complete under the induced metric, i.e., if every Cauchy sequence is convergent.

Thus, the terms "Banach space" and "complete normed space" are interchangeable.

An important fact that we will assume without proof is that the complex plane $\mathbb{C}$ under absolute value is a Banach space.

A.3.1 Infinite Series in Normed Spaces

Since a normed space has both an operation of vector addition and a notion of convergence, we can consider infinite series in normed spaces.

Definition A.10 (Convergent Series). Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in a normed linear space X. Then the series $\sum_{n=1}^{\infty} f_n$ *converges and equals* $f \in X$ if the *partial sums* $s_N = \sum_{n=1}^{N} f_n$ converge to f, i.e., if

$$\lim_{N\to\infty} \|f - s_N\| = \lim_{N\to\infty} \left\| f - \sum_{n=1}^{N} f_n \right\| = 0.$$

Note that the ordering of a series may be important! If we reorder a series, or in other words consider a new series $\sum_{n=1}^{\infty} f_{\sigma(n)}$ where $\sigma\colon \mathbb{N} \to \mathbb{N}$ is a bijection, there is no guarantee that this reordered series will still converge. These issues are addressed in more detail in Section A.11.

Next we give one sufficient condition for convergence of a series.

Definition A.11 (Absolutely Convergent Series). Let X be a normed space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X. If

$$\sum_{n=1}^{\infty} \|f_n\| < \infty,$$

then we say that the series $\sum_{n=1}^{\infty} f_n$ is *absolutely convergent* in X.

The definition of absolute convergence does not itself tell us that the series $\sum f_n$ converges in X. That will always true if X is a Banach space, but it need not be true if X is not complete.

Exercise A.12. Let X be a Banach space and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X. Prove that if $\sum \|f_n\| < \infty$ then the series $\sum f_n$ converges in X.

Moreover, in this case the convergence is <u>unconditional</u>, i.e., regardless of the ordering of the series

The converse of this exercise is also true, and is often a useful method for proving that a given normed space is a Banach space.

Exercise A.13. Let X be a normed space. Prove that X is a Banach space if and only if every absolutely convergent series in X converges in X.

A.3.2 Convexity

Definition A.14 (Convex Set). If X is a vector space and $K \subseteq X$, then K is *convex* if

$$x, y \in K,\ 0 \le t \le 1 \implies tx + (1-t)y \in K.$$

Thus, the entire line segment between x and y is contained in K (including the midpoint $\frac{1}{2}x + \frac{1}{2}y$ in particular).

Every subspace of a vector space is convex by definition. The fact that balls in a normed space are convex is an important property.

Exercise A.15. Show that if X is a normed linear space, then each open ball $B_r(f)$ in X is convex.

Additional Problems

A.1. If $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a normed space X and there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ that converges to $f \in X$, then $f_n \to f$.

A.2. If $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a normed space X, then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N}, \quad \|f_{n_{k+1}} - f_{n_k}\| < 2^{-k}.$$

A.3. If $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a normed space X and if we have $\|f_{n+1} - f_n\| < 2^{-n}$ for every n, then $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy.

A.4. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in a normed space X, and let $f \in X$ be fixed. Suppose that every subsequence $\{g_n\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence $\{h_n\}_{n\in\mathbb{N}}$ such that $h_n \to f$. Show that $f_n \to f$.

<u>Example</u>

The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is a series in the Banach space $\mathbb{C}$ that converges, but does not converge absolutely.

Exercise hints

Exercises from Appendix A

A.12 Hint: Let $s_N = \sum_{n=1}^{N} f_n$, and show that the sequence of partial sums $\{s_N\}_{N\in\mathbb{N}}$ is Cauchy in X.

A.13 Hints: Suppose that every absolutely convergent series is convergent. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Show that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $\|f_{n_{k+1}} - f_{n_k}\| < 2^{-k}$ for every k (see Problem A.2). Then $\sum_k (f_{n_{k+1}} - f_{n_k})$ is absolutely convergent, hence converges, say to f. Show that $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence that converges (consider the partial sums of $\sum_k (f_{n_{k+1}} - f_{n_k})$). Show $\{f_n\}_{n\in\mathbb{N}}$ converges (consider Problem A.1).