The ℓ^p Spaces

A.4 Examples of Banach Spaces: ℓ^p, $\cancel{C_b, C_0, C_b^m}$

In this section we give a few examples of Banach spaces and complete metric spaces.

A.4.1 The ℓ^p Spaces

We begin with the ℓ^p spaces on countable index sets. Compare the following definition with the definition of of L^p given later in Definition B.44.

Definition A.16. Let I be a finite or countably infinite index sequence.

(a) If $0 < p < \infty$, then $\ell^p(I)$ consists of all sequences of scalars $x = (x_k)_{k\in I}$ such that

$$\|x\|_p \;=\; \|(x_k)_{k\in I}\|_p \;=\; \left(\sum_{k\in I} |x_k|^p\right)^{1/p} \;<\; \infty.$$

(b) For $p = \infty$, the space $\ell^\infty(I)$ consists of all sequences of scalars $x = (x_k)_{k\in I}$ such that

$$\|x\|_\infty \;=\; \|(x_k)_{k\in I}\|_\infty \;=\; \sup_{k\in I} |x_k| \;<\; \infty.$$

If $I = \mathbb{N}$, then we write simply ℓ^p instead of $\ell^p(\mathbb{N})$.

If $I = \{1, \dots, d\}$, then $\ell^p(I) = \mathbb{C}^d$, and in this case we refer to $\ell^p(I)$ as "$\mathbb{C}^d$ under the ℓ^p norm." The ℓ^2 norm on $\mathbb{C}^d$ is called the *Euclidean norm.*

The following inequality is extremely important (recall the definition of the dual index p' introduced in Section A.1).

Theorem A.17 (Hölder's Inequality). *Let I be a finite or countable index set. Given $1 \le p \le \infty$, if $x = (x_k)_{k\in I} \in \ell^p(I)$ and $y = (y_k)_{k\in I} \in \ell^{p'}(I)$, then $xy = (x_k y_k)_{k\in I} \in \ell^1(I)$, and*

$$\|xy\|_1 \le \|x\|_p \, \|y\|_{p'}.$$

For $1 < p < \infty$, this inequality is

$$\sum_{k\in I} |x_k y_k| \;\le\; \left(\sum_{k\in I} |x_k|^p\right)^{1/p} \left(\sum_{k\in I} |y_k|^{p'}\right)^{1/p'}.$$

Proof. The cases $p = 1$ and $p = \infty$ are straightforward exercises. Assume $1 < p < \infty$. The key to the proof is a special case of *Young's Inequality* for continuous, strictly increasing functions (note that this is a different inequality than Young's Convolution Inequality that is presented in Exercise 1.23). Namely, because x^{p-1} is continuous and strictly increasing and because $y^{\frac{1}{p-1}}$ is its inverse function, for all a, $b \ge 0$ we have

Note $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) \in \ell^p$ for $1 < p \le \infty$.

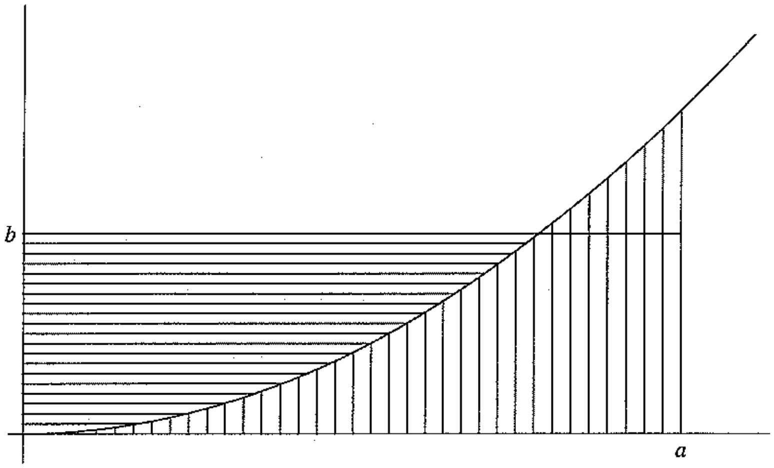

Fig. A.1. Illustration of Young's Inequality. Area of the vertically hatched region is $\int_0^a x^{p-1}\,dx$; area of the horizontally hatched region is $\int_0^b y^{\frac{1}{p-1}}\,dy$; area of the rectangle is ab.

$$ab \le \int_0^a x^{p-1}\,dx + \int_0^b y^{\frac{1}{p-1}}\,dy = \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

Q. When does equality hold?

(see the "proof by picture" in Figure A.1, or see Problem A.8).

Consequently, if $x \in \ell^p(I)$ and $y \in \ell^{p'}(I)$ satisfy $\|x\|_p = 1 = \|y\|_{p'}$, then

$$\|xy\|_1 = \sum_{k\in I} |x_k y_k| \le \sum_{k\in I} \left(\frac{|x_k|^p}{p} + \frac{|y_k|^{p'}}{p'}\right) = \frac{1}{p} + \frac{1}{p'} = 1. \qquad \text{(A.2)}$$

For general nonzero x, y, we apply (A.2) to the normalized vectors $x/\|x\|_p$ and $y/\|y\|_{p'}$ to obtain

$$\frac{\|xy\|_1}{\|x\|_p\,\|y\|_{p'}} = \left\|\frac{x}{\|x\|_p}\,\frac{y}{\|y\|_{p'}}\right\|_1 \le 1. \qquad \square$$

The next exercise shows that if $p \ge 1$ then $\|\cdot\|_p$ is a norm on $\ell^p(I)$. The Triangle Inequality on ℓ^p (often called *Minkowski's Inequality*) is easy to prove for $p = 1$ and $p = \infty$, but more difficult for $1 < p < \infty$. A hint for using Hölder's Inequality to prove Minkowski's Inequality is given in the solutions section at the end of the text.

Exercise A.18. Let I be a finite or countable index set. Show that if $1 \le p \le \infty$, then $\|\cdot\|_p$ is a norm on $\ell^p(I)$, and $\ell^p(I)$ is a Banach space with respect to this norm.

~~On the other hand, if $p < 1$ then $\|\cdot\|_p$ fails the Triangle Inequality and hence is not a norm. Still, we can modify the distance function so that ℓ^p is~~

~~so $(1-t)g+th \in B_r(f)$, and therefore $B_r(f)$ is convex.~~

Proof:

A.18 Fix $1 < p < \infty$. To show that the Triangle Inequality holds, choose x, $y \in \ell^p(I)$, and write

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{k\in I} |x_k+y_k|^{p-1}\,|x_k+y_k| \\
&\le \sum_{k\in I} |x_k+y_k|^{p-1}\,|x_k| \;+\; \sum_{k\in I} |x_k+y_k|^{p-1}\,|y_k| \\
&\le \biggl(\sum_{k\in I}\bigl(|x_k+y_k|^{p-1}\bigr)^{p'}\biggr)^{1/p'} \biggl(\sum_{k\in I}|x_k|^p\biggr)^{1/p} \\
&\qquad + \biggl(\sum_{k\in I}\bigl(|x_k+y_k|^{p-1}\bigr)^{p'}\biggr)^{1/p'} \biggl(\sum_{k\in I}|y_k|^p\biggr)^{1/p} \\
&= \biggl(\sum_{k\in I}|x_k+y_k|^p\biggr)^{(p-1)/p} \biggl(\sum_{k\in I}|x_k|^p\biggr)^{1/p} \\
&\qquad + \biggl(\sum_{k\in I}|x_k+y_k|^p\biggr)^{(p-1)/p} \biggl(\sum_{k\in I}|y_k|^p\biggr)^{1/p} \\
&= \|x+y\|_p^{p-1}\,\|x\|_p + \|x+y\|_p^{p-1}\,\|y\|_p,
\end{aligned}$$

where we have applied Hölder's Inequality and used the fact $p' = p/(p-1)$. Dividing both sides by $\|x+y\|_p^{p-1}$ therefore yields $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

To show completeness, fix $0 < p \le \infty$, and suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in ℓ^p. Write $x_n = (x_n(1), x_n(2), \dots)$. Then for each fixed k we have $|x_m(k) - x_n(k)| \le \|x_m - x_n\|_p$, so $\{x_n(k)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of scalars. Define $x(k) = \lim_{n\to\infty} x_n(k)$.

Now choose $\varepsilon > 0$. Then there exists an N such that $\|x_m - x_n\|_p < \varepsilon$ for all m, $n > N$. Fix $n > N$, and consider first the case $p < \infty$. For each $M > 0$ we have

$$\sum_{k=1}^{M} |x(k)-x_n(k)|^p \;=\; \lim_{m\to\infty} \sum_{k=1}^{M} |x_m(k)-x_n(k)|^p \;\le\; \lim_{m\to\infty} \|x_m-x_n\|_p^p \;\le\; \varepsilon^p.$$

Since this is true for every M, we conclude that for all $n > N$ we have

$$\|x-x_n\|_p^p \;=\; \sum_{k=1}^{\infty} |x(k)-x_n(k)|^p \;\le\; \varepsilon^p.$$

Consequently, $\|x\|_p \le \|x-x_n\|_p + \|x_n\|_p < \infty$, so $x \in \ell^p$ and we also have $x_n \to x$ in ℓ^p. This shows that ℓ^p is complete for all $p < \infty$.

For $p = \infty$, we likewise have for $n > N$ and all $k \in \mathbb{N}$ that

$$|x(k) - x_n(k)| \;=\; \lim_{m\to\infty} |x_m(k) - x_n(k)| \;\le\; \|x_m - x_n\|_\infty \;\le\; \varepsilon,$$

so $\|x - x_n\|_\infty \le \varepsilon$ for all $n > N$. As before, it follows that $x \in \ell^\infty$ and $x_n \to x$ in ℓ^∞.

On the other hand, if $p < 1$ then $\|\cdot\|_p$ fails the Triangle Inequality and hence is not a norm. Still, we can modify the distance function so that ℓ^p is at least a complete metric space, though this metric is not induced by any norm. ← Exercise

Exercise A.19. Let I be a finite or countably infinite index set. Show that if $0 < p < 1$, then $\|x+y\|_p^p \le \|x\|_p^p + \|y\|_p^p$. Consequently, $\ell^p(I)$ is a vector space, and $\mathrm{d}(x,y) = \|x-y\|_p^p$ is a metric on $\ell^p(I)$. Show that $\ell^p(I)$ is complete with respect to this metric. However, if I contains more than one element, then the unit ball $B_1(0)$ is not convex, and hence this metric is not induced from any norm (compare Exercise A.15).

We can also define $\ell^p(I)$ when the index set I is uncountable. In this case, for $p < \infty$ we define $\ell^p(I)$ to be the space of all sequences $x = (x_k)_{k\in I}$ with at most countably many terms nonzero and such that $\sum |x_k|^p < \infty$. With this definition, $\ell^p(I)$ is again a Banach space for $1 \le p \le \infty$.

Proof:

A.19 Suppose that $0 < p < 1$. Let $f(t) = (1+t)^p$ and $g(t) = 1 + t^p$ for $t > 0$. Then $f(0) = 1 = g(0)$. Also,

$$f'(t) \;=\; p\,(1+t)^{p-1} \;=\; p\,\frac{1}{(1+t)^{1-p}} \qquad \text{and} \qquad g'(t) \;=\; pt^{p-1} \;=\; p\,\frac{1}{t^{1-p}}.$$

Since $0 < 1 - p < 1$, we have $t^{1-p} < (1+t)^{1-p}$, and therefore $f'(t) \le g'(t)$ for $t > 0$. Hence g is increasing faster than f, and therefore $f(t) \le g(t)$ for all $t \ge 0$. Next, given any a, $b \ge 0$, we have

$$(a+b)^p \;=\; a^p\left(1+\frac{b}{a}\right)^p \;\le\; a^p\left(1+\left(\frac{b}{a}\right)^p\right) \;=\; a^p + b^p.$$

Hence, if x, $y \in \ell^p(I)$, then

$$\|x+y\|_p^p \;=\; \sum_{k\in I} |x_k + y_k|^p \;\le\; \sum_{k\in I} \left(|x_k|^p + |y_k|^p\right) \;=\; \|x\|_p^p + \|y\|_p^p.$$

This establishes the Triangle Inequality. Completeness follows very similarly to the proof for the case $p \ge 1$.

Additional Problems

A.5. Fix $0 < p \le \infty$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of vectors in $\ell^p(I)$, and x a vector in $\ell^p(I)$. Write the components of x_n and x as $x_n = (x_n(1), x_n(2), \dots)$ and $x = (x(1), x(2), \dots)$.

(a) Show that if $x_n \to x$ in $\ell^p(I)$, then x_n *converges componentwise* to x, i.e., for each fixed k we have $\lim_{n\to\infty} x_n(k) = x(k)$.

(b) If I is finite then componentwise convergence implies convergence with respect to the norm $\|\cdot\|_p$.

(c) If I is infinite then componentwise convergence need not imply convergence in the norm of $\ell_p(I)$.

Note this inclusion: for $1 \le p \le \infty$, ℓ^1 is the smallest space, and ℓ^∞ the largest.

A.6. If $1 \le p < q \le \infty$, then $\ell^p \subsetneq \ell^q$, and $\|x\|_q \le \|x\|_p$ for all $x \in \ell^p$.

To show strict inclusion, consider $x_k = (k \log^2 k)^{-1/q}$

A.7. Let I be a finite or countable index set, and let $w\colon I \to (0,\infty)$ be fixed. Given a sequence of scalars $x = (x_k)_{k\in I}$, set

$$\|x\|_{p,w} = \begin{cases} \left(\sum\limits_{k\in I} |x_k|^p\, w(k)^p\right)^{1/p}, & 0 < p < \infty, \\ \sup\limits_{k\in I} |x_k|\, w(k), & p = \infty, \end{cases}$$

and define the *weighted* ℓ^p *space*

$$\ell^p_w(I) = \Big\{x : \|x\|_p < \infty\Big\}.$$

Show that $\ell^p_w(I)$ is a Banach space for each $1 \le p \le \infty$.

A.8. (a) Show that if $0 < \theta < 1$, then $t^\theta \le \theta t + (1-\theta)$ for $t > 0$, with equality if and only if $t = 1$.

(b) Suppose that $a, b \ge 0$. Apply part (a) with $t = a^p b^{-p'}$ and $\theta = 1/p$ to show that $ab \le a^p/p + b^{p'}/p'$, with equality if and only if $b = a^{p-1}$.

A.9. Show that equality holds in Hölder's Inequality (Theorem A.17) if and only if there exist scalars α, β, not both zero, such that $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ for each $k \in I$.

A.10. Show that if $0 < p < 1$ and I contains at least two elements, then the unit ball in $\ell^p(I)$ is not convex. Conclude that there is <u>no norm</u> on $\ell^p(I)$ that induces the metric $d(x,y) = \|x-y\|_p^p$.