A.7.3 Equivalent Norms

Now we consider the equivalence of convergence criteria and topologies for the specific case of normed spaces.

Definition A.53. Suppose that X is a normed linear space with respect to a norm $\|\cdot\|_a$ and also with respect to another norm $\|\cdot\|_b$. Then we say that these norms are *equivalent* if there exist constants C_1, $C_2 > 0$ such that

$$\forall f \in X, \quad C_1 \|f\|_a \leq \|f\|_b \leq C_2 \|f\|_a. \tag{A.4}$$

We write $\|\cdot\|_a \asymp \|\cdot\|_b$ to denote that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms.

Theorem A.54. *Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on a vector space X. Then the following statements are equivalent.*

(a) $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *are equivalent norms.*

(b) $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *induce the same topologies on* X. ← So they give the same open sets.

(c) $\|\cdot\|_a$ *and* $\|\cdot\|_b$ *define the same convergence criterion. That is, if* $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence in* X *and* $x \in X$, *then*

$$\lim_{n\to\infty} \|x - x_n\|_a = 0 \iff \lim_{n\to\infty} \|x - x_n\|_b = 0.$$

Proof. (a) $\Rightarrow$ (c). This is immediate from the definition of equivalent norms.

(c) $\Rightarrow$ (b). This follows from Exercise A.48.

(b) $\Rightarrow$ (a). Assume that statement (b) holds. Let $B_r^a(x)$ and $B_r^b(x)$ denote the open balls of radius r centered at $x \in X$ with respect to $\|\cdot\|_a$ and $\|\cdot\|_b$, respectively. Since $B_1^a(0)$ is open with respect to $\|\cdot\|_a$, statement (b) implies that $B_1^a(0)$ is open with respect to $\|\cdot\|_b$. Therefore, since $0 \in B_1^a(0)$, there must exist some $r > 0$ such that $B_r^b(0) \subseteq B_1^a(0)$.

Now choose any $x \in X$ and any $\varepsilon > 0$. Then

$$\frac{(r-\varepsilon)}{\|x\|_b}\, x \in B_r^b(0) \subseteq B_1^a(0),$$

so

$$\left\| \frac{(r-\varepsilon)\, x}{\|x\|_b} \right\|_a < 1.$$

Rearranging, this implies $(r-\varepsilon)\,\|x\|_a < \|x\|_b$. Since this is true for every ε, we conclude that $r\,\|x\|_a \leq \|x\|_b$.

A symmetric argument, interchanging the roles of the two norms, shows that there exists an $s > 0$ such that $\|x\|_b \leq s\,\|x\|_a$ for every $x \in X$. Hence the two norms are equivalent. □

Given any finite-dimensional vector space X, we can define many norms on X, analogous to the ℓ^p norms defined in Section A.4.

Exercise A.55. Let $\mathcal{B} = \{x_1, \dots, x_d\}$ be any basis for a finite-dimensional vector space X, and let $x = \sum_{k=1}^{d} c_k(x)\, x_k$ denote the unique expansion of $x \in X$ with respect to this basis (the vector $[x]_\mathcal{B} = (c_1(x), \dots, c_d(x))$ is called the *coordinate vector* of x with respect to the basis $\mathcal{B}$). Show that

$$\|x\|_p = \begin{cases} \left(\displaystyle\sum_{k=1}^{d} |c_k(x)|^p \right)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\max_k |c_k(x)|, & p = \infty, \end{cases}$$

are norms on X, and that X is complete with respect to each of these norms. Note that $\|x\|_p$ is simply the ℓ^p norm of the coordinate vector $[x]_\mathcal{B}$.

Although we will not prove it, the following is an important fact regarding norms on finite-dimensional spaces.

Theorem A.56. *If X is a finite-dimensional vector space, then any two norms on X are equivalent. In particular, if $\|\cdot\|$ is any norm on X and $\|\cdot\|_p$ is any one of the norms constructed in Exercise A.55, then $\|\cdot\| \asymp \|\cdot\|_p$.*

<u>Problem</u>

Show that all of the norms in Exercise A.55 are equivalent. (Give a direct proof.)