A.8 Closed and Dense Sets

The smallest closed set that contains a given set is called its closure, defined precisely as follows.

Definition A.57. If E is a subset of a topological space X, then the *closure of* E, denoted $\overline{E}$, is the smallest closed set in X that contains E:

$$\overline{E} = \bigcap\{F \subseteq X : F \text{ is closed and } F \supseteq E\}.$$

This is the abstract definition

If $\overline{E} = X$, then we say that E is *dense* in X.

Often it is more convenient to use the following equivalent form of the closure of a set.

Exercise A.58. Given a subset E of a topological space X, show that $\overline{E}$ is the union of E and all the accumulation points of E.

There are many different notations and terminology that are commonly used when discussing subspaces of a normed space. In particular, some authors make the restriction that the term "subspace" is reserved to mean a "closed subspace." Other authors use the term "linear manifold" to denote a subspace that need not be closed. To avoid ambiguity, a subspace for us will mean a subspace in the usual vector space sense, i.e., a subset that is closed under both vector addition and scalar multiplication. We will refer to a subspace that is also a closed set as a *closed subspace.*

The typical method for showing that a subset of a metric space is dense is given in the next exercise.

We will take this as our definition of dense set.

Exercise A.59. Let X be a metric space, and let $E \subseteq X$ be given. Show that E is dense in X if and only if for each $f \in X$ there exist a sequence $\{f_n\}_{n\in\mathbb{N}}$ in E such that $f_n \to f$.

In a finite-dimensional normed space, every subspace is a closed set. The following exercises illustrate that this need not be the case in infinite dimensions.

Exercise A.60. (a) Fix $1 \le p \le \infty$. Prove that

$$c_{00} = \{x = (x_1, \dots, x_N, 0, 0, \dots) : N > 0,\, x_1, \dots, x_N \in \mathbb{C}\}$$

is a subspace of ℓ^p that is not closed (with respect to the ℓ^p-norm). Prove that c_{00} is dense in $\ell^p(\mathbb{N})$ if $p < \infty$, but that it is not dense in ℓ^∞. The vectors in c_{00} are sometimes called *finite sequences* because they contain at most finitely many nonzero components.

(b) Define

$$c_0 = \Big\{x = (x_k)_{k=1}^\infty : \lim_{k\to\infty} x_k = 0\Big\}.$$

Prove c_0 is a closed subspace of $\ell^\infty(\mathbb{N})$, and that c_0 is the closure of c_{00} in the ℓ^∞-norm.

Exercise A.61. Show that the space $C_c(\mathbb{R})$ introduced in equation (A.3) is a dense subspace of $C_0(\mathbb{R})$ that is not closed (under the uniform norm).

The significance of closed subspaces is given in the following exercise.

Exercise A.62. Let M be a subspace of a Banach space X. Then M is itself a Banach space (using the norm inherited from X) if and only if M is closed.

Hence, c_{00} is a normed space that is not complete with respect to any norm $\|\cdot\|_p$, $1 \le p \le \infty$. Similarly, $C_c(\mathbb{R})$ is a normed space that is not complete with respect to the uniform norm (compare Exercise A.21).

We now introduce a definition that in some sense distinguishes between "small" and "large" infinite-dimensional spaces.

Definition A.63. A topological space that contains a countable dense subset is said to be *separable*.

Exercise A.64. (a) Show that if I is a finite or countable index set, then $\ell^p(I)$ is separable for $1 \le p < \infty$. Show that if I is infinite, then $\ell^\infty(I)$ is not separable.

(b) Show that $C_0(\mathbb{R})$ is separable. $\leftarrow$ A little tricky.

Additional Problems

A.19. Show that every finite-dimensional subspace of a normed linear space is closed.

Solution. sketch

A.58 Solution sketch. Let A be the union of E and the accumulation points of E, and suppose that $x \notin A$. Then x is not an accumulation point of E, so there exists an open neighborhood U of x such that $E \cap (U \backslash \{x\}) = \emptyset$. Since $x \notin E$, this implies U contains no points of E. Show that U cannot contain any accumulation points of E either, and conclude that $U \subseteq X \backslash A$. Therefore $X \backslash A$ is open, so A is closed, and consequently $\overline{E} \subseteq A$.

Hint.

A.64 Hints: (b) Let θ_M be 1 on $[-M, M]$, zero outside $[-M-1, M+1]$, and linear on $[-M-1, -M]$ and $[M, M+1]$. Use the Weierstrass Approximation Theorem (Theorem A.77) to show that

$$S = \left\{ \sum_{k=0}^{N} r_k x^k \theta_M(x) : M \in \mathbb{N},\, N \geq 0,\, \mathrm{Re}\,(r_k),\, \mathrm{Im}\,(r_k) \in \mathbb{Q} \right\}$$

is countable and dense in $C_0(\mathbb{R})$.

Problem **A.19** Solution sketch. Let M be a finite-dimensional subspace of a normed space X. Suppose that $f_n \in M$ and $f_n \to g \in X$. If $g \notin M$, define

$$M_1 = M + \operatorname{span}\{g\} = \{m + cg : m \in M,\, c \in \mathbb{C}\}.$$

Show that if $f = m + cg$ with $m \in M$ and $c \in \mathbb{C}$, then $\|f\|_{M_1} = \|m\| + |c|$ is a well-defined norm on M_1. By Theorem A.56, all norms on M_1 are equivalent, so $f_n \to g$ in the norm of M_1. But $\|g - f_n\|_{M_1} = \|f_n\| + 1 \geq 1$ for every n, so this is a contradiction.