A.14 Urysohn's Lemma

Urysohn's Lemma is a general result that holds in a large class of topological spaces (specifically, the *normal* topological spaces, which include all metric spaces). It states that if A and B are disjoint closed subsets of a normal topological space X, then there exists a continuous function $f\colon X \to [0,1]$ that is identically 0 on A and identically 1 on B. We prove here a version of Urysohn's Lemma for $\mathbb{R}^d$ (although the same simple proof can be used in any metric space). A more refined version of Urysohn's Lemma for real line is proved in Chapter 1 (see Theorem 1.58).

First, we need the following lemma.

Lemma A.102. *If $E \subseteq \mathbb{R}^d$ is nonempty, then*

$$f(x) \;=\; \operatorname{dist}(x,E) \;=\; \inf\bigl\{|x-z| : z \in E\bigr\}$$

is uniformly continuous on $\mathbb{R}^d$.

Proof. Fix $\varepsilon > 0$, and set $\delta = \varepsilon/2$. Choose any x, $y \in \mathbb{R}^d$ with $|x-y| < \varepsilon/2$. By definition, there exist a, $b \in E$ such that

$$|x-a| \;<\; \operatorname{dist}(x,E) + \frac{\varepsilon}{2} \qquad \text{and} |y-b| \;<\; \operatorname{dist}(y,E) + \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned} f(y) \;=\; \operatorname{dist}(y,A) \;&\le\; |y-a| \\ &\le\; |y-x| + |x-a| \\ &<\; \frac{\varepsilon}{2} + \operatorname{dist}(x,E) + \frac{\varepsilon}{2} \\ &=\; f(x) + \varepsilon. \end{aligned}$$

Similarly $f(x) < f(y) + \varepsilon$, so $|f(x) - f(y)| < \varepsilon$. ◻

Theorem A.103 (Urysohn's Lemma). *If E, F are disjoint closed subsets of $\mathbb{R}^d$, then there exists a continuous function $\theta\colon \mathbb{R}^d \to \mathbb{R}^d$ such that*

(a) $0 \le \theta \le 1$,

(b) $\theta = 0$ *on* E, *and*

(c) $\theta = 1$ *on* F.

Proof. Because E is closed, if $x \notin E$ then we have that $\operatorname{dist}(x,E) > 0$. Since $\operatorname{dist}(x,E)$ and $\operatorname{dist}(x,F)$ are continuous functions of x by Lemma A.102, it follows that the function

$$\theta(x) \;=\; \frac{\operatorname{dist}(x,E)}{\operatorname{dist}(x,E) + \operatorname{dist}(x,F)}$$

has the required properties. ◻