See also Section 6.1 in Folland.

B.6 The L^p Spaces

In this section we introduce and examine the spaces $L^p(E)$, which are Banach spaces for $1 \le p \le \infty$, and complete metric spaces for $0 < p < 1$.

B.6.1 Norm and Completeness

Definition B.43 (Integrable Function). Let $E \subseteq \mathbb{R}^d$ be measurable. Then a measurable function f on E (either extended real-valued or complex-valued) is *integrable* on E if $\int_E |f| < \infty$.

Note that any integrable function must be finite almost everywhere.

These are the definitions of the L^p spaces with respect to Lebesgue measure. Analogous definitions can be made for any positive measure μ on a measure space (X, Σ).

Definition B.44. Let $E \subseteq \mathbb{R}^d$ be measurable.

(a) If $0 < p < \infty$, then $L^p(E)$ consists of all measurable functions $f\colon E \to \mathbb{C}$ such that $|f|^p$ is integrable, i.e.,

$$\|f\|_p = \left(\int_E |f|^p\right)^{1/p} < \infty.$$

(b) For $p = \infty$, the space $L^\infty(E)$ consists of all those measurable functions $f\colon E \to \mathbb{C}$ for which $|f|$ is essentially bounded, i.e.,

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in E} |f(x)| < \infty.$$

We refer to $L^p(\mathbb{R})$ for $p < \infty$ as the *Lebesgue space of p-integrable functions*, and to $L^\infty(\mathbb{R})$ as the *Lebesgue space of essentially bounded functions.*

The proof of Hölder's Inequality for ℓ^p, given in Theorem A.17, carries over to $L^p(E)$ (we use the notation for the dual index p' introduced in Section A.1).

Exercise B.45 (Hölder's Inequality). Let $E \subseteq \mathbb{R}$ be measurable, and fix $1 \le p \le \infty$. If $f \in L^p(E)$ and $g \in L^{p'}(E)$ then $fg \in L^1(E)$, and

$$\|fg\|_1 \le \|f\|_p \, \|g\|_{p'}.$$

For $1 < p < \infty$, this inequality is

$$\int_E |fg| \le \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^{p'}\right)^{1/p'}.$$

Similarly to Problem A.9, equality holds in Hölder's Inequality if and only if there exist scalars α, β, not both zero, such that $\alpha\,|f(x)|^p = \beta\,|g(x)|^{p'}$ a.e.

In Exercise A.18 we saw that the ℓ^p spaces are Banach spaces for $p \ge 1$. An analogous result holds for $L^p(E)$, but the fact that $\int_E |f|^p = 0$ only implies that $f = 0$ a.e. adds the technical complication that $\|\cdot\|_p$ is only a seminorm and not a norm on $L^p(E)$. We will deal with this issue now.

Hölder for $p = p' = 2$ is called Cauchy-Schwarz.

Exercise B.46. If $E \subseteq \mathbb{R}^d$ be measurable and $1 \le p \le \infty$, then $\|\cdot\|_p$ is a seminorm on $L^p(E)$.

The triangle inequality on L^p is also known as *Minkowski's Inequality.*

Unfortunately, if E is any set of measure zero then $\|\chi_E\|_p = 0$ even though χ_E is not the zero function. On the other hand, $\chi_E = 0$ a.e., which suggests that when dealing with the L^p spaces we may not wish to distinguish between functions that are equal almost everywhere. Indeed, the next exercise is the standard procedure for "converting" a seminorm into a norm by forming equivalence classes.

Exercise B.47. Show that the relation $f \sim g$ if $f = g$ a.e. is an equivalence relation on $L^p(E)$. Let $\widetilde{f}$ denote the equivalence class of f in $L^p(E)$ under this relation, and set $|||\widetilde{f}|||_p = \|f\|_p$. Show that this quantity is independent of the choice of representative f. Let the quotient space $\widetilde{L^p}(E)$ consist of all the distinct equivalence classes of $f \in L^p(E)$, i.e.,

$$\widetilde{L^p}(E) = \{\widetilde{f} : f \in L^p(E)\}.$$

Show that $\widetilde{L^p}(E)$ is a normed space with respect to $|||\cdot|||_p$.

Typically we abuse notation and let the symbol f denote the equivalence class $\widetilde{f}$ of all functions equal to f a.e., and we write $L^p(E)$ instead of $\widetilde{L^p}(E)$. In other words, we identify any two functions that are equal a.e. Adopting this convention, the completeness of the L^p spaces can be proved. We state this here, and assign the proof as an exercise in the next section (see Exercise B.61).

Theorem B.48. *Let $E \subseteq \mathbb{R}^d$ be measurable.*

(a) *If $1 \le p \le \infty$, then $\|\cdot\|_p$ is a norm on $L^p(E)$, and $L^p(E)$ is a Banach space with respect to this norm.*

(b) *If $0 < p < 1$, then $\mathrm{d}(f,g) = \|f-g\|_p^p$ is a metric on $L^p(E)$, and $L^p(E)$ is complete with respect to this metric.*

The proof that $\|\cdot\|_p$ for $1 \le p \le \infty$ is a norm, or that $\|\cdot\|_p^p$ for $0 < p < 1$ is a metric, is exactly similar to the proof for ℓ^p, & is left as an exercise. The real issue is completeness.

Proof of Completeness of $L^p(E)$.

B.61 Assume that $0 \le p < \infty$ and that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(E)$. Similarly to Exercise B.60, it follows from Tchebyshev's Inequality that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure. Theorem B.29 therefore implies that there exists a measurable function f on E such that $f_n \xrightarrow{m} f$. Hence, by Theorem B.28, there exist f_{n_k} such that $f_{n_k}(x) \to f(x)$ for a.e. $x \in E$.

Choose $\varepsilon > 0$. Then there exists an N such that

$$j, k > N \implies \|f_{n_j} - f_{n_k}\|_p^p < \varepsilon.$$

Fix $j > N$. Since $f_{n_k}(x) \to f(x)$ pointwise a.e., we have by Fatou's Lemma that

$$\|f - f_{n_k}\|_p^p \le \lim_{j\to\infty} \int_E |f_{n_j}(x) - f_{n_k}(x)|^p \, dx \le \varepsilon.$$

Hence $\|f\|_p \le \|f - f_{n_k}\|_p + \|f_{n_k}\|_p < \infty$, so $f \in L^p(E)$ and furthermore $f_{n_k} \to f$ in $L^p(E)$. Thus, $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence that has a subsequence that converges to f, so by Problem A.1 we know that $f_n \to f$ in $L^p(E)$.

Now consider the case $p = \infty$. Assume $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(E)$. For each m, $n \in \mathbb{N}$, set

$$Z_{mn} = \{x \in E : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\}.$$

Then $Z = \cup_{m,n} Z_{mn}$ has measure zero. For each $x \in E \backslash Z$, we have

$$|f_m(x) - f_n(x)| \le \|f_m - f_n\|_\infty,$$

so $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of scalars. This sequence therefore converges, so for $x \in E\backslash Z$ define $f(x) = \lim_{n\to\infty} f_n(x)$. Since each f_n is measurable and since $|Z| = 0$, it follows from Exercise B.23 that f is measurable.

Now choose $\varepsilon > 0$. Then there exists an N such that $\|f_m - f_n\|_\infty \le \varepsilon$ for all m, $n > N$. Hence for any $x \in E\backslash Z$, we have

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \le \lim_{m\to\infty} \|f_m - f_n\|_\infty \le \varepsilon.$$

Since this is true for all x in E except for a set of measure zero, we conclude that $\|f - f_n\|_\infty \le \varepsilon$ for all $n > N$. As before, this implies $f \in L^\infty(E)$ and $f_n \to f$ in $L^\infty(E)$.

Alternative proof for $1 \le p < \infty$. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is an absolutely convergence series in $L^p(\mathbb{R})$, say

$$B = \sum_{n=1}^{\infty} \|f_n\|_p < \infty.$$

Set

$$g_N(x) = \sum_{n=1}^{N} |f_n(x)| \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} |f_n(x)|.$$

B.6.2 On Abuses of Notation

Ignoring the distinction between a function and the equivalence class of functions that are equal to it a.e. is not usually a problem, but on occasion some care needs to be taken, as when dealing with a continuous function. Every function in $C_b(\mathbb{R})$ is continuous and bounded. Therefore we write $C_b(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, but in doing so we are really identifying $C_b(\mathbb{R})$ with its image in $L^\infty(\mathbb{R})$ under the equivalence relation $\sim$, i.e., if $f \in C_b(\mathbb{R})$ then it determines an equivalence class $\widetilde{f}$ of functions in $L^\infty(\mathbb{R})$ that are equal to it almost everywhere. Conversely, if we are given $f \in L^\infty(\mathbb{R})$ (really an equivalence class $\widetilde{f}$ of functions), then the statement $f \in C_b(\mathbb{R})$ means that there is a representative of this equivalence class that belongs to $C_b(\mathbb{R})$.

Note that the two statements "f is continuous a.e." and "f equals a continuous function a.e." are distinct. The first means that $\lim_{y\to x} f(y) = f(x)$ for almost every x, while the second means that there exists a continuous function g such that $f(x) = g(x)$ for almost every x (see Problem B.7).

Exercise B.50. Show that if $f \in C_b(\mathbb{R})$, then

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| \;=\; \sup_{x\in\mathbb{R}} |f(x)|.$$

Consequently, for continuous bounded functions, the uniform norm defined in Exercise A.21 coincides with the L^∞-norm defined above.

Another place where the fact that elements of $L^p(E)$ are equivalence classes must be taken into account is when discussing the support of a function in $L^p(E)$. For example, $\chi_\mathbb{Q}$ is one representative of the zero function in $L^p(\mathbb{R})$, yet the support of $\chi_\mathbb{Q}$ is the entire real line. Unfortunately, the support of a function depends very much on the choice of representative. Still, it is such a convenient concept that we usually abuse notation and apply support terminology to elements of $L^p(E)$. For example, we write "f has compact support" with the understanding that this means that some representative of f has compact support, or, in other words, there is a compact set K such that $f(x) = 0$ for almost every $x \notin K$.

Exercise B.51. Suppose that $\mathrm{T} \subseteq \mathbb{R}$ is closed. Show that if we follow the convention given above, then

$$\operatorname{supp}(f) \subseteq \mathrm{T} \iff f(x) = 0 \text{ for a.e. } x \notin \mathrm{T}.$$

The space $L^2(E)$ is special. As before, we consider elements of $L^2(E)$ to be equivalence classes of functions that are equal almost everywhere.

Exercise B.49. If $E \subseteq \mathbb{R}^d$ is measurable, show that

$$\langle f, g \rangle = \int_E f(x)\, \overline{g(x)}\, dx$$

defines an inner product on $L^2(E)$, and $L^2(E)$ is a Hilbert space with respect to this inner product.

We'll return to Q3 later

B.6.3 Local Integrability

We won't need this very much

We introduce one final space in this section. This space is not a Banach space, but it is very useful when we only need to consider integrability at a "local" level.

Definition B.52 (Locally Integrable Functions). *locally integrable function* The space of *locally integrable functions* on $\mathbb{R}^d$ is

$$L^1_{\mathrm{loc}}(\mathbb{R}^d) \;=\; \bigl\{f\colon \mathbb{R}^d \to \mathbb{C} : f \cdot \chi_K \in L^1(\mathbb{R}) \text{ for every compact } K \subseteq \mathbb{R}^d\bigr\}.$$

function!locally integrable

As we do for the L^p spaces, we regard the elements of $L^1_{\mathrm{loc}}(\mathbb{R})$ as being equivalence classes of functions that are equal almost everywhere. $L^1_{\mathrm{loc}}(\mathbb{R})$ is not a Banach space, but it is a good example of a topological vector space whose topology is defined by an infinite family of seminorms, see Example E.5 in the Appendices.

We can likewise define $L^p_{\mathrm{loc}}(\mathbb{R})$. Note that

$$L^p(\mathbb{R}) \;\subsetneq\; L^p_{\mathrm{loc}}(\mathbb{R}) \;\subsetneq\; L^1_{\mathrm{loc}}(\mathbb{R}), \qquad 1 \le p \le \infty.$$

$L^1_{\mathrm{loc}}(\mathbb{R})$ contains many functions that do not belong to any $L^p(\mathbb{R})$. For example, every polynomial belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$, as does e^{x^2}.

Note that this inclusion is the reverse of the one for ℓ^p: when $|E|<\infty$, $L^\infty(E)$ is the "small space" and $L^1(E)$ is the "large space"

Additional Problems

B.9. Show that $C_0(\mathbb{R})\backslash L^1(\mathbb{R}) \neq \emptyset$, and $(L^1(\mathbb{R}) \cap C(\mathbb{R}))\backslash C_b(\mathbb{R}) \neq \emptyset$.

B.10. Show that if $|E| < \infty$ and $0 < p \le q \le \infty$, then $L^q(E) \subseteq L^p(E)$. In contrast, show that $L^p(\mathbb{R})$ is not contained in $L^q(\mathbb{R})$ for any p and q.

No inclusions when $|E|=\infty$!

B.11. Show that equality holds in Hölder's Inequality (Exercise B.45) if and only if there exist scalars α, β, not both zero, such that $\alpha\,|f|^p = \beta\,|g|^{p'}$ a.e.

B.12. Prove that if $|E| < \infty$, then $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

B.13. Given $1 \le p < q \le \infty$, show that $L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ is a Banach space under the norm $\|f\| = \|f\|_p + \|f\|_q$. Further, if $1 \le p < r < q \le \infty$ then we have $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subseteq L^r(\mathbb{R})$, and if $r < \infty$ then $L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ is dense in $L^r(\mathbb{R})$.

B.14. Given $1 \le p < q \le \infty$, show that

$$L^p(\mathbb{R}) + L^q(\mathbb{R}) \;=\; \{f + g : f \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})\}$$

is a Banach space under the norm

$$\|f\| \;=\; \inf\{\|g\|_p + \|h\|_q : f = g + h \text{ with } g \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})\}.$$

Further, if $1 \le p < r < q \le \infty$ then we have $L^r(\mathbb{R}) \subseteq L^p(\mathbb{R}) + L^q(\mathbb{R})$.

B.15. Define $e_n(x) = e^{2\pi i n x}$. Show that $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal sequence in $L^2[0,1]$.

Remark: It is shown in Section 1.11 that this this sequence is complete in $L^2[0,1]$, and hence forms an orthonormal basis for $L^2[0,1]$.

B.16. Generalize Hölder's Inequality to the case of more than two functions. Show that if $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = 1$ and $f_i \in L^{p_i}(\mathbb{R})$ for $i = 1, \dots, k$, then $f_1 \cdots f_k \in L^1(\mathbb{R})$ and

$$\|f_1 \cdots f_k\|_1 \;\le\; \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

Problem. Show that if $0 < p < 1$, then $\|\cdot\|_p$ does not satisfy the Triangle Inequality on $L^p[0,1]$.

Problem. $x^\alpha \in L^p(0,1)$ for what α?
$x^\alpha \in L^p(1,\infty)$ for what α?

Since these are nonnegative series, they converge pointwise to a value in $[0, \infty]$. By the Triangle Inequality, for each N we have

$$\|g_N\|_p \le \sum_{n=1}^{N} \|f_n\|_p \le B.$$

Since $|g_N|^p \nearrow |g|^p$, we have by the Monotone Convergence Theorem that

$$\|g\|_p^p = \int |g|^p = \lim_{N\to\infty} \int |g_N|^p = \lim_{N\to\infty} \|g_N\|_p^p \le B.$$

Therefore $g \in L^p(\mathbb{R})$. By Fubini's Theorem, it follows that the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges pointwise a.e. Since $|f| \le g$, we have $f \in L^p(\mathbb{R})$. Also, if we set

$$h_N(x) = \sum_{n=1}^{N} f_n(x),$$

then $h_N \to f$ pointwise a.e., and

$$|f - h_N|^p \le \big(|f| + |h_N|\big)^p \le \big(g + g_N\big)^p \le \big(2g\big)^p \in L^1(\mathbb{R}).$$

Therefore $h_N \to f$ in $L^p(\mathbb{R})$ by the Lebesgue Dominated Convergence Theorem. But this exactly says that $f = \sum_{n=1}^{\infty} f_n$ converges in $L^p(\mathbb{R})$. Hence every absolutely convergent series in $L^p(\mathbb{R})$ is convergent, and therefore $L^p(\mathbb{R})$ is complete.

Problem: Tchebyshev's Inequality

Fix $0<p<\infty$ & $E \subseteq \mathbb{R}$ (measurable).

Show that if $f \in L^p(E)$ & $\alpha > 0$, then

$$|\{|f|>\alpha\}| \le \frac{1}{\alpha^p} \int |f|^p = \frac{1}{\alpha^p} \|f\|_p^p$$

(See Folland, Section 6.3)

Problem Hints

B.10 Hint: To show $L^p(E) \subseteq L^q(E)$ when $|E| < \infty$ and $p < q$, apply Hölder's Inequality to $\int_E |f|^p \cdot 1$ using exponents q/p and $(q/p)'$.

To show that $L^p(\mathbb{R})$ is never contained in $L^q(\mathbb{R})$, consider functions of the form $x^\alpha \cdot \chi_{(0,1)}(x)$ or $x^\alpha \cdot \chi_{(1,\infty)}(x)$.

B.13 Hint: To show the inclusion, define $s = \frac{q-p}{q-r}$ and $t = \frac{q-p}{r-p}$, and observe that $\frac{1}{s} + \frac{1}{t} = 1$ and $\frac{p}{s} + \frac{q}{t} = r$.

B.14 Hint: To show completeness, suppose that $\sum f_n$ is an absolutely convergent series in the norm $\|\cdot\|$. Then there exist $g_n \in L^p(\mathbb{R})$ and $h_n \in L^q(\mathbb{R})$ such that $f_n = g_n + h_n$ and $\|g_n\|_p + \|h_n\|_q < 2^{-n}$. Use the completeness of $L^p(\mathbb{R})$ to show that $g = \sum g_n$ converges in L^p-norm, and similarly $h = \sum h_n$ converges in L^q-norm.

For the inclusion, given $f \in L^r(\mathbb{R})$, consider $g = f \cdot \chi_{\{|f|>1\}}$ and $h = f \cdot \chi_{\{|f|\leq 1\}}$.