Convergence in L^p

B.7 Convergence Theorems

If $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions that converge pointwise almost everywhere to a function f, then it need not be true that $\lim_{n\to\infty} \int_E f_n$ will converge to $\int_E f$ (see Exercise B.57 below). In this section we review several important theorems related to this issue.

The following result is also known as the *Beppo Levi Theorem.* We say that a sequence of extended real-valued functions $\{f_n\}_{n\in\mathbb{N}}$ is *monotone increasing* if $f_1(x) \le f_2(x) \le \cdots$ for all x. We write $f_n \nearrow f$ to denote that $\{f_n\}_{n\in\mathbb{N}}$ is monotone increasing and that $f_n(x) \to f(x)$ pointwise.

Theorem B.53 (Monotone Convergence Theorem). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a monotone increasing sequence of measurable, nonnegative functions on a measurable set $E \subseteq \mathbb{R}^d$, and define $f(x) = \lim_{n\to\infty} f_n(x)$, so $f_n \nearrow f$. Then*

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Note that $R(f_1, E) \subseteq R(f_2, E) \subseteq \cdots$ and that $R(f, E) = \cup R(f_n, E)$. Exercise B.10(d) therefore implies that $|R(f_n, E)|$ converges to $|R(f, E)|$. □

Since changing the value of a function on a set of zero measure does not change the value of its integral, it suffices to assume that the hypotheses in the Monotone Convergence Theorem hold a.e. instead of everywhere, and the same is true of the other theorems in this section.

As an application of the Monotone Convergence Theorem, we can now complete the proof of Theorem B.35. The proof relies on the useful fact that we can always find simple functions that increase pointwise to an arbitrary nonnegative measurable function.

Review

Exercise B.54. Suppose that $E \subseteq \mathbb{R}^d$ and $f, g \colon E \to [0, \infty]$ are measurable.

(a) Show that

$$\phi_n(x) = \begin{cases} \dfrac{j-1}{2^n}, & \text{if } \dfrac{j-1}{2^n} \le f(x) < \dfrac{j}{2^n},\ j = 1, \dots, n2^n, \\ n, & \text{if } f(x) \ge n. \end{cases}$$

is a simple function, $\phi_n(x) \nearrow f(x)$ for each x, and if f is bounded then ϕ_n converges uniformly to f. Apply the Monotone Convergence Theorem to conclude that $\int_E \phi_n \nearrow \int_E f$.

(b) Prove Theorem B.35, i.e., show that $\int_E (f+g) = \int_E f + \int_E g$.

Since the partial sums of a series of nonnegative functions form a monotone increasing sequence, we have another application of the Monotone Convergence Theorem, this time to the problem of interchanging a sum and an integral. This is a special case of the abstract version of Tonelli's Theorem, compare Theorems B.65 and D.70.

Corollary B.55. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable, nonnegative functions on a measurable set $E \subseteq \mathbb{R}^d$. Then*

$$\int_E \left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} \int_E f_n.$$

If the functions f_n are nonnegative but are not monotone increasing, then we may not be able to interchange a limit and an integral. However, the following result states that if the functions f_n are all nonnegative, then we do at least have a particular inequality.

Exercise B.56 (Fatou's Lemma). If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable, nonnegative functions on measurable set $E \subseteq \mathbb{R}^d$, then

$$\int_E \left(\liminf_{n\to\infty} f_n\right) \le \liminf_{n\to\infty} \int_E f_n.$$

Exercise B.57. Give examples showing that strict inequality can hold in Fatou's Lemma.

Review

Now we come to the workhorse of the stable of convergence theorems.

Exercise B.58 (Lebesgue Dominated Convergence Theorem). Assume $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of measurable functions on measurable set $E \subseteq \mathbb{R}^d$, either extended real-valued or complex-valued, such that:

(a) $f(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. $x \in E$, and

(b) there exists $g \in L^1(E)$ such that $|f_n(x)| \le g(x)$ a.e. for every n.

Then f_n converges to f in L^1-norm, i.e.,

$$\lim_{n\to\infty} \|f - f_n\|_1 = \lim_{n\to\infty} \int_E |f - f_n| = 0,$$

and, consequently,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Let us make some general observations concerning convergence in $L^p(E)$. By definition, if p is finite then $f_n \to f$ in $L^p(E)$ means that

$$\lim_{n\to\infty} \int_E |f(x) - f_n(x)|^p \, dx = 0.$$

Hence, there will surely be a need to use the convergence theorems presented above when dealing with convergence in L^p, but most likely applied to $|f - f_n|^p$ rather than to f_n and f.

In the ℓ^p spaces, convergence in ℓ^p norm implies componentwise convergence (see Problem A.5). The situation in $L^p(E)$ is a little different. For $p = \infty$, if $f_n \to f$ in $L^\infty(E)$ then it follows that $f_n(x) \to f(x)$ pointwise a.e. However, for p finite, an L^p-convergent sequence need not converge pointwise.

Exercise B.59. Let $0 < p < \infty$ be fixed. Give an example of functions $f_n \in L^p(\mathbb{R})$ such that $f_n \to 0$ in L^p-norm (i.e., $\|f_n\|_p \to 0$), but $f_n(x)$ does not converge pointwise to zero as $n \to \infty$.

Fortunately, it is true that an L^p-convergent sequence always has a subsequence that converges pointwise almost everywhere.

Exercise B.60. Let $E \subseteq \mathbb{R}^d$ be measurable and fix $0 < p \leq \infty$. Show that if $f_n, f \in L^p(E)$ and $f_n \to f$ in $L^p(E)$, then $f_n \xrightarrow{\mathrm{m}} f$. Consequently, there exists a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ such that $f_{n_k}(x) \to f(x)$ for almost every $x \in E$.

We end this section with some useful properties of the L^p spaces. First, we show that they are Banach spaces.

We did this earlier.

Exercise B.61. Let $E \subseteq \mathbb{R}$ be Lebesgue measurable and fix $0 < p < \infty$. Use Fatou's Lemma and Exercise B.60 to prove that $L^p(E)$ is complete (Theorem B.48). Show also that $L^\infty(E)$ is complete.

It is often useful to know that we can approximate a given L^p function by functions that have some special properties. For example, combining the Lebesgue Dominated Convergence Theorem with Exercise B.54, we see that the L^p simple functions are dense. In fact, we can even restrict further to simple functions with compact support.

Exercise B.62. Let $E \subseteq \mathbb{R}^d$ be Lebesgue measurable, and show that the set S consisting of all compactly supported simple functions is dense in $L^p(E)$ for each $1 \leq p < \infty$.

Do this!

We can then use the denseness of the simple functions to prove the very useful fact that the space of continuous, compactly supported functions is dense in $L^p(\mathbb{R}^d)$. For the case $p = \infty$, we have instead that $C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$ in the uniform norm.

Theorem B.63. *$C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for each $1 \leq p < \infty$.*

Proof. First consider the function $f = \chi_E$ where $E \subseteq \mathbb{R}^d$ is bounded. If we fix $\varepsilon > 0$, then there exists a bounded open set $U \supseteq E$ such that $|U \backslash E| < \varepsilon$. Also, by Problem B.4, there exists a compact set $K \subseteq E$ such that $|E \backslash K| < \varepsilon$. By Urysohn's Lemma (Theorem A.103), we can find a continuous function $\theta \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $0 \leq \theta \leq 1$, $\theta = 1$ on K, and $\theta = 0$ on $\mathbb{R}^d \backslash U$. Then $\theta \in C_c(\mathbb{R}^d)$, and we have

$$\|\chi_E - \theta\|_p^p = \int |\chi_E - \theta|^p = \int_{U \backslash K} |\chi_E - \theta|^p \leq |U \backslash K| \leq \varepsilon.$$

Hence χ_E can be approximated arbitrarily closely in L^p-norm by elements of $C_c(\mathbb{R}^d)$.

Exercise: Complete the proof by making use of Exercise B.62. □

Using Theorem B.63, we can show that the space of finite linear combinations of intervals (sometimes called "really simple functions") is dense in $L^p(\mathbb{R})$. This provides us with another useful set of approximating functions for $L^p(\mathbb{R})$.

Exercise B.64. Show that $\{\chi_{[a,b]} : -\infty < a < b < \infty\}$ is a complete set in $L^p(\mathbb{R})$ when $1 \le p < \infty$.

Additional Problems

B.17. Let $f \in L^1(\mathbb{R}^d)$ be given. If $\varepsilon > 0$, then there exists a $\delta > 0$ such that if $E \subseteq \mathbb{R}^d$ is measurable and $|E| < \delta$, then $\int_E |f| < \varepsilon$. In particular, if $|E| = 0$, then $\int_E f = 0$.

B.18. Show that $L^p(\mathbb{R})$ is separable for $1 \le p < \infty$, but $L^\infty(\mathbb{R})$ is not separable.

Problem Hints

B.17 Hint: Define $E_n = \{x \in \mathbb{R}^d : |f(x)| \leq n\}$ and show that $f_n = f\chi_{E_n} \to f$ in $L^1(\mathbb{R}^d)$.

B.18 Hint: The set of all intervals $[a, b]$ with rational endpoints is countable. Show that

$$S = \left\{ \sum_{k=1}^{n} r_k \, \chi_{[a_k, b_k]} : n \in \mathbb{N},\ \mathrm{Re}\,(r_k),\ \mathrm{Im}\,(r_k) \in \mathbb{Q},\ a_k < b_k \in \mathbb{Q} \right\}$$

is countable and dense in $L^p(\mathbb{R})$ when $1 \leq p < \infty$.