See also Section 5.5 in Folland.

A.5 Inner Products

While a norm on a vector space provides a notion of the length of a vector, an inner product additionally provides us with a notion of the angle between vectors.

Definition A.23 (Semi-Inner Product, Inner Product). If H is a vector space over the complex field $\mathbb{C}$, then a *semi-inner product* on H is a function $\langle\cdot,\cdot\rangle\colon H \times H \to \mathbb{C}$ such that for all $f, g, h \in H$ and scalars $\alpha, \beta \in \mathbb{C}$ we have:

(a) $\langle f,f\rangle \ge 0$,

(b) $\langle f,g\rangle = \overline{\langle g,f\rangle}$,

(c) Linearity in the first variable: $\langle \alpha f + \beta g, h\rangle = \alpha\langle f,h\rangle + \beta\langle g,h\rangle$.

If a semi-inner product $\langle\cdot,\cdot\rangle$ also satisfies:

(d) $\langle f,f\rangle = 0$ if and only if $f = 0$,

then it is called an *inner product* on H. In this case, H is called a *inner product space* or a *pre-Hilbert space*.

Semi-inner products are known by many names. For example, a semi-inner product is sometimes called a *positive Hermitian form* on H (compare Problem A.10). There are many different standard notations for semi-inner products, including $[f,g]$, (f,g), or $\langle f|g\rangle$, in addition to our preferred notation $\langle f,g\rangle$.

Exercise A.24. If $\langle\cdot,\cdot\rangle$ is a semi-inner product on a vector space H, show that the following statements hold.

(a) Antilinearity in the second variable: $\langle f, \alpha g + \beta h\rangle = \bar{\alpha}\langle f,g\rangle + \bar{\beta}\langle f,h\rangle$.

(b) $\langle f,0\rangle = 0 = \langle 0,f\rangle$.

(c) $\langle 0,0\rangle = 0$.

A function of two variables that is linear in the first variable and antilinear (also called *conjugate linear*) in the second variable *conjugate linear function* is referred to as a *sesquilinear form*. Thus each semi-inner product $\langle\cdot,\cdot\rangle$ is an example of a sesquilinear form. The prefix "sesqui-" means "one and a half."

The next exercise gives the prototypical example of an inner product.

Exercise A.25. If I is a finite or countably infinite index set, show that

$$\Big\langle (x_k)_{k\in I},\, (y_k)_{k\in I} \Big\rangle \;=\; \sum_{k\in I} x_k\,\overline{y}_k$$

defines an inner product on $\ell^2(I)$.

Other texts on Hilbert spaces include Gohberg/Goldberg & Debnath/Mikusinski.

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Every subspace S of an inner product space H is itself an inner product space (using the inner product on H restricted to S).

Our next goal is to show that every semi-inner product induces a seminorm on H, and every inner product induces a norm.

Notation A.26. If $\langle\cdot,\cdot\rangle$ is a semi-inner product on a vector space H, then we write

$$\|f\| \;=\; \langle f,f\rangle^{1/2}, \quad f\in H.$$

Prejudicing the issue, we refer to $\|\cdot\|$ as the *seminorm induced by* $\langle\cdot,\cdot\rangle$, and if $\langle\cdot,\cdot\rangle$ is an inner product, then we call $\|\cdot\|$ the *norm induced by* $\langle\cdot,\cdot\rangle$.

Before showing that $\|\cdot\|$ actually is a seminorm (or norm), we derive some of its basic properties.

Exercise A.27. Given a semi-inner product $\langle\cdot,\cdot\rangle$ on a vector space H, show that the following statements hold for all $f,\, g\in H$.

(a) Polar Identity: $\|f+g\|^2 = \|f\|^2 + 2\operatorname{Re}\langle f,g\rangle + \|g\|^2$.
(b) Parallelogram Law: $\|f+g\|^2 + \|f-g\|^2 = 2\big(\|f\|^2+\|g\|^2\big)$.

Now we prove an important inequality; this should be compared to Hölder's Inequality (Theorem A.17) for the particular case $p=2$. This inequality is variously known as the *Schwarz*, *Cauchy–Schwarz*, or *Cauchy–Bunyakowski–Schwarz Inequality.*

Theorem A.28 (Cauchy–Bunyakowski–Schwarz Inequality). *If $\langle\cdot,\cdot\rangle$ is a semi-inner product on a vector space H, then*

$$\forall\, f,g,\in H, \quad |\langle f,g\rangle| \;\le\; \|f\|\,\|g\|.$$

Proof. If $f=0$ or $g=0$ then there is nothing to prove, so suppose both are nonzero. Write $\langle f,g\rangle = \alpha\,|\langle f,g\rangle|$ where $\alpha\in\mathbb{C}$ and $|\alpha|=1$. Then for $t\in\mathbb{R}$ we have by the Polar Identity that

$$\begin{aligned} 0 \;\le\; \|f-\alpha t g\|^2 \;&=\; \|f\|^2 - 2\operatorname{Re}\bar{\alpha}t\langle f,g\rangle + t^2\,\|g\|^2 \\ &=\; \|f\|^2 - 2t\,|\langle f,g\rangle| + t^2\,\|g\|^2. \end{aligned}$$

This is a real-valued quadratic polynomial in the variable t. In order for it to be nonnegative, it can have at most one real root. This requires that the discriminant be at most zero, so

$$(-2\,|\langle f,g\rangle|)^2 - 4\,\|f\|^2\,\|g\|^2 \;\le\; 0.$$

The desired inequality then follows upon rearranging. □

When $\langle f,g\rangle = 0$, we say that f and g are *orthogonal.* More details on orthogonality appear in Section A.12.

Finally, the Cauchy–Bunyakowski–Schwarz Inequality can be used to show that $\|\cdot\|$ is indeed a seminorm or norm on H.

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Exercise A.29. Given a semi-inner product $\langle \cdot, \cdot \rangle$ on a vector space H, show that $\| \cdot \|$ is a seminorm on H, and that if $\langle \cdot, \cdot \rangle$ is an inner product then $\| \cdot \|$ is a norm on H.

A.29 The only property that is not obvious is the Triangle Inequality. For this, note that by the Polar Identity and the Cauchy–Bunyakowski–Schwarz Inequality, we have

$$\begin{aligned}\|f+g\|^2 = \langle f+g, f+g\rangle &= \|f\|^2 + 2\,\mathrm{Re}\,\langle f,g\rangle + \|g\|^2 \\ &\le \|f\|^2 + 2\,|\langle f,g\rangle| + \|g\|^2 \\ &\le \|f\|^2 + 2\,\|f\|\,\|g\| + \|g\|^2 \\ &= \left(\|f\| + \|g\|\right)^2.\end{aligned}$$

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Thus, all inner product spaces are normed linear spaces, and hence we can use all the notions of convergence introduced in earlier sections. On the other hand, the following exercise shows that given an arbitrary normed space, there need not be an inner product on the space that induces that norm.

Exercise A.30. Let I be a finite or countably infinite index set containing at least two elements. Show that if $1 \le p \le \infty$ with $p \ne 2$, then $\|\cdot\|_p$ does not satisfy the Parallelogram Law. Therefore, there is no inner product on $\ell^p(I)$ whose induced norm is $\|\cdot\|_p$.

On the other hand, it is certainly possible for $\ell^p(I)$ to be an inner product space with respect to *some* inner product. For example, if I is finite or if $p < 2$, then $\ell^p(I) \subseteq \ell^2(I)$, so in this case $\ell^p(I)$ is an inner product space with respect to the inner product on $\ell^2(I)$ restricted to the subspace $\ell^p(I)$. However, but the norm induced from this inner product is $\|\cdot\|_2$ and not $\|\cdot\|_p$. Further, if I is infinite then $\ell^p(I)$ is not complete with respect to the induced norm $\|\cdot\|_2$, whereas it is complete with respect to the norm $\|\cdot\|_p$.

Definition A.31 (Hilbert Space). An inner product space H is called a *Hilbert space* if it is complete with respect to the induced norm, i.e., if every Cauchy sequence is convergent.

Thus, a Hilbert space is an inner product that is a Banach space with respect to the induced norm, or, in other words, it is a complete inner product space. In particular, $\ell^2(I)$ is a Hilbert space.

Another example is $L^2(E)$ for $E \subseteq \mathbb{R}$.

Additional Problems

A.10. Show that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathbb{C}^d$ if and only if there exists a positive definite matrix A such that $\langle x, y\rangle = Ax \cdot y$, where $x \cdot y = x_1\bar{y}_1 + \cdots + x_d\bar{y}_d$ denotes the usual dot product on $\mathbb{C}^d$.

A.11. Continuity of the inner product: Show that if H is an inner product space and $f_n \to f$, $g_n \to g$ in H, then $\langle f_n, g_n\rangle \to \langle f, g\rangle$.

A.12. Let H be an inner product space. Show that if the series $\sum_{n=1}^{\infty} f_n$ converges in H, then for any $g \in H$,

$$\left\langle \sum_{n=1}^{\infty} f_n, g \right\rangle = \sum_{n=1}^{\infty} \langle f_n, g\rangle.$$

Note that this is not merely a consequence of the linearity of the inner product in the first variable — the continuity of the inner product is also needed.

A.13. Show that equality holds in the Cauchy–Bunyakowski–Schwarz Inequality if and only if there exist scalars $\alpha, \beta \in \mathbb{C}$, not both zero, such that $\|\alpha f + \beta g\| = 0$. In particular, if $\|\cdot\|$ is a norm, then either $f = cg$ or $g = cf$ for some scalar c.

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A.14. Suppose that X is a Banach space whose norm $\|\cdot\|$ satisfies the Parallelogram Law. Prove that

$$\langle f, g\rangle \;=\; \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right)$$

is an inner product on X, and the original norm $\|\cdot\|$ is the induced norm, i.e., $\|f\|^2 = \langle f, f\rangle$ for every $f \in X$.

A.15. Justify the following statement: The angle between two vectors f, g in a Hilbert space H is the value of θ that satisfies $\operatorname{Re}\langle f, g\rangle = \|f\|\,\|g\|\cos\theta$.