A.12 Orthogonality

In this section we review some of the special properties of inner product spaces, especially in relation to orthogonality.

Definition A.83. Let H be an inner product space, and let I be an arbitrary index set.

(a) Vectors $f, g \in H$ are *orthogonal*, denoted $f \perp g$, if $\langle f, g\rangle = 0$.

(b) A collection of vectors $\{f_i\}_{i\in I}$ is *orthogonal* if $\langle f_i, f_j\rangle = 0$ whenever $i \neq j$.

(c) A collection of vectors $\{f_i\}_{i\in I}$ is *orthonormal* if it is orthogonal and each vector is a unit vector, i.e.,

$$\langle f_i, f_j\rangle \;=\; \delta_{ij} \;=\; \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

For example, the standard basis is an orthonormal sequence in ℓ^2.

The first important property of orthogonal vectors is the *Pythagorean Theorem*.

Exercise A.84 (Pythagorean Theorem). Show that if $f_1, \dots, f_n \in H$ are orthogonal, then

$$\Bigl\|\sum_{k=1}^{n} f_k\Bigr\|^2 \;=\; \sum_{k=1}^{n} \|f_k\|^2.$$

The existence of the notion of orthogonality gives inner product spaces a much "simpler" structure than general Banach spaces. We will derive below some of the basic properties of inner product spaces, most of which are straightforward consequences of orthogonality. Some (but not all) of these results have analogues for general Banach spaces. However, even for the results that do have analogues, the corresponding proofs in the non-Hilbert space are usually much more complicated or are nonconstructive.

A useful notion in a Hilbert space is the orthogonal direct sum of subspaces.

Definition A.85 (Orthogonal Direct Sum). Let M, N be closed subspaces of a Hilbert space H.

(a) The *direct sum* of M and N is

$$M + N \;=\; \{f + g : f \in M, g \in N\}.$$

(b) We say that M and N are *orthogonal subspaces*, denoted $M \perp N$, if $f \perp g$ for every $f \in M$ and $g \in N$.

(c) If M, N are orthogonal subspaces in H, then we call their direct sum the *orthogonal direct sum* of M and N, and denote it by $M \oplus N$.

Exercise A.86. Show that if M, N are closed, orthogonal subspaces of H, then $M \oplus N$ is a closed subspace of H.

A.12.1 Orthogonal Projections and Orthogonal Complements

Definition A.87. Let X be a normed linear space and let $A \subseteq H$. The *distance* from a point $f \in H$ to the set A is

$$\operatorname{dist}(f, A) \;=\; \inf\{\|f-g\| : g \in A\}.$$

A useful property is that given any closed convex subset K of a Hilbert space H and given any $x \in H$, there is a unique point in K that is closest to x.

Theorem A.88 (Closest Point Property). *If H is a Hilbert space and K is a nonempty closed, convex subset of H, then given any $h \in H$ there exists a unique point $k_0 \in K$ that is closest to h. That is, there is a unique point $k_0 \in K$ such that*

$$\|h-k_0\| \;=\; \operatorname{dist}(h,K) \;=\; \inf\{\|h-k\| : k \in K\}.$$

Proof. Set $d = \operatorname{dist}(h,K) = \inf\{\|h-k\| : k \in K\}$. By definition, there exist $k_n \in K$ such that $\|h-k_n\| \to d$, and furthermore we have $d \le \|h-k_n\|$ for each n. Therefore, if we fix any $\varepsilon > 0$ then we can find an N such that

$$n > N \quad \Longrightarrow \quad d^2 \le \|h-k_n\|^2 \le d^2+\varepsilon^2.$$

By the Parallelogram Law (Exercise A.27),

$$\|(h-k_n)-(h-k_m)\|^2 + \|(h-k_n)+(h-k_m)\|^2 \;=\; 2\left(\|h-k_n\|^2+\|h-k_m\|^2\right).$$

Hence,

$$\begin{aligned}\left\|\frac{k_m-k_n}{2}\right\|^2 &= \frac{1}{4}\|(h-k_n)-(h-k_m)\|^2\\ &= \frac{\|h-k_n\|^2}{2}+\frac{\|h-k_m\|^2}{2}-\left\|h-\frac{k_m+k_n}{2}\right\|^2.\end{aligned}$$

However, $\frac{k_m+k_n}{2} \in K$ since K is convex, so $\|h-\frac{k_m+k_n}{2}\| \ge d$. Also, if m, $n > N$ then $\|h-k_n\|^2$, $\|h-k_m\|^2 \le d^2+\varepsilon^2$. Therefore,

$$\left\|\frac{k_m-k_n}{2}\right\|^2 \;\le\; \frac{d^2+\varepsilon^2}{2}+\frac{d^2+\varepsilon^2}{2}-d^2 \;=\; \varepsilon^2.$$

So, $\|k_m-k_n\| \le 2\varepsilon$ for all m, $n > N$, which says that the sequence $\{k_n\}_{k\in\mathbb{N}}$ is Cauchy. Since H is complete, this sequence must converge, i.e., $k_n \to k_0$ for some $k_0 \in H$. But $k_n \in K$ for all n and K is closed, so we must have $k_0 \in K$. Since $h-k_n \to h-k_0$, we have

$$\|h-k_0\| \;=\; \lim_{n\to\infty}\|h-k_n\| \;=\; d,$$

and thus $\|h-k_0\| \le \|h-k\|$ for every $k \in K$. Thus k_0 is a closest point in K to h, and we leave as an exercise the task of proving that k_0 is unique. □

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Applying the closest point theorem to the particular case of closed subspaces gives us the existence of *orthogonal projections* in a Hilbert space.

Theorem A.89. *Let M be a closed subspace of a Hilbert space H. If $h \in H$, then the following statements are equivalent.*

(a) *$h = p + e$ where p is the (unique) point in M closest to h.*

(b) *$h = p + e$ where $p \in M$ and $e \perp M$ (i.e., $e \perp f$ for every $f \in M$).*

Proof. (a) $\Rightarrow$ (b). Let p be the (unique) point in M closest to h, and let $e = p - h$. Choose any $f \in M$. We must show that $\langle f, e\rangle = 0$. Since M is a subspace, $p + \lambda f \in M$ for any scalar $\lambda \in \mathbb{C}$. Hence,

$$\begin{aligned} \|h-p\|^2 \;\le\; \|h-(p+\lambda f)\|^2 \;&=\; \|(h-p)-\lambda f\|^2 \\ &=\; \|h-p\|^2 - 2\,\mathrm{Re}\,(\langle \lambda f, h-p\rangle) + |\lambda|^2\,\|f\|^2 \\ &=\; \|h-p\|^2 - 2\,\mathrm{Re}\,(\lambda\langle f, e\rangle) + |\lambda|^2\,\|f\|^2. \end{aligned}$$

Therefore,

$$\forall\,\lambda \in \mathbb{C}, \quad 2\,\mathrm{Re}\,(\lambda\langle f,e\rangle) \;\le\; |\lambda|^2\,\|f\|^2.$$

If we consider $\lambda = t > 0$, then we can divide through by t to get

$$\forall\, t > 0, \quad 2\,\mathrm{Re}\,(\langle f,e\rangle) \;\le\; t\,\|f\|^2.$$

Letting $t \to 0^+$, we conclude that $\mathrm{Re}\,(\langle f,e\rangle) \le 0$. Similarly, taking $\lambda = t < 0$ and letting $t \to 0^-$, we obtain $\mathrm{Re}\,(\langle f,e\rangle) \ge 0$, so $\mathrm{Re}\,(\langle f,e\rangle) = 0$.

Finally, by taking $\lambda = it$ with $t > 0$ and then $\lambda = it$ with $t < 0$, it follows similarly that $\mathrm{Im}\,(\langle f,e\rangle) = 0$.

(b) $\Rightarrow$ (a). Exercise: Apply the Pythagorean Theorem. □

Definition A.90 (Orthogonal Projection). Let M be a closed subspace of a Hilbert space H. For each $h \in H$, the point $p \in M$ closest to H is called the *orthogonal projection* of h onto M.

Definition A.91 (Orthogonal Complement). Let A be a subset (not necessarily a subspace) of a Hilbert space H. The *orthogonal complement* of A is

$$A^\perp \;=\; \{f \in H : f \perp A\} \;=\; \{f \in H : \langle f, g\rangle = 0 \text{ for all } g \in A\}.$$

In the terminology of orthogonal complements, a vector p is the orthogonal projection of f onto a closed subspace M if and only if $f = p + e$ where $p \in M$ and $e \in M^\perp$.

Exercise A.92. Show that if A is an arbitrary subset of a Hilbert space H, then $A^\perp$ is always a closed subspace of H (even if A is not), and $(A^\perp)^\perp = \overline{\mathrm{span}}(A)$.

In particular, if M is a closed subspace of a Hilbert space H then we have $(M^\perp)^\perp = M$. Further, H is the orthogonal direct sum of M and $M^\perp$:

$$H = M \oplus M^\perp.$$

The next exercise will allow us to give a useful equivalent formulation of completeness of a sequence in a Hilbert space.

Exercise A.95. Let A be a subset of a Hilbert space H. Show that A is complete if and only if $A^\perp = \{0\}$, i.e., the only vector orthogonal to every element of A is the zero vector.