A.13 Orthogonality and Complete Sequences

In a Hilbert space, the combination of completeness and orthonormality of a sequence $\{e_n\}_{n\in\mathbb{N}}$ leads to especially nice series representations of vectors in H in terms of the vectors e_n. We will explore this topic in this section.

First, the following exercise summarizes some basic results connected to convergence of series of orthonormal vectors.

Exercise A.96. Let $\{e_n\}_{n\in\mathbb{N}}$ be any orthonormal sequence in a Hilbert space H. Show that the following statements hold.

(a) Bessel's Inequality: $\sum_{n=1}^{\infty} |\langle f, e_n\rangle|^2 \le \|f\|^2$.

(b) If $f = \sum_{n=1}^{\infty} c_n e_n$ converges, then $c_n = \langle f, e_n\rangle$.

(c) $\sum_{n=1}^{\infty} c_n e_n$ converges $\iff$ $\sum_{n=1}^{\infty} |c_n|^2 < \infty$.

(d) If $\sum_{n=1}^{\infty} c_n e_n$ converges then it converges *unconditionally*, i.e., it converges regardless of the ordering of the indices (see Section A.11).

(e) $f \in \overline{\text{span}}(\{e_n\}_{n\in\mathbb{N}}) \iff f = \sum_{n=1}^{\infty} \langle f, e_n\rangle\, e_n$.

(f) If $f \in H$, then $p = \sum_{n=1}^{\infty} \langle f, e_n\rangle\, e_n$ is the orthogonal projection of f onto $\overline{\text{span}}(\{e_n\}_{n\in\mathbb{N}})$.

Proof.

A.96 (a) Choose $f \in H$. For each N define

$$f_N = f - \sum_{n=1}^{N} \langle f, e_n \rangle e_n.$$

If $1 \le k \le N$, then we have

$$\langle f_N, e_k \rangle = \langle f, e_k \rangle - \sum_{n=1}^{N} \langle f, e_n \rangle \langle e_n, e_k \rangle = \langle f, e_k \rangle - \langle f, e_k \rangle = 0.$$

Thus $f_N \perp e_1, \dots, e_N$. Therefore, by the Pythagorean Theorem, we have

$$\begin{aligned} \|f\|^2 &= \Big\| f_N + \sum_{n=1}^{N} \langle f, e_n \rangle e_n \Big\|^2 \\ &= \|f_N\|^2 + \sum_{n=1}^{N} \|\langle f, e_n \rangle e_n\|^2 \\ &= \|f_N\|^2 + \sum_{n=1}^{N} |\langle f, e_n \rangle|^2 \ \ge\ \sum_{n=1}^{N} |\langle f, e_n \rangle|^2. \end{aligned}$$

Letting $N \to \infty$, we obtain Bessel's inequality.

(c) $\Leftarrow$. Suppose that $\sum_{n=1}^{\infty} |c_n|^2 < \infty$. Set

$$s_N = \sum_{n=1}^{N} c_n e_n, \qquad t_N = \sum_{n=1}^{N} |c_n|^2.$$

We know that $\{t_N\}_{N\in\mathbb{N}}$ is a convergent (hence Cauchy) sequence of scalars, and we must show that $\{s_N\}_{N\in\mathbb{N}}$ is a convergent sequence of vectors. We have for $N > M$ that

$$\begin{aligned} \|s_N - s_M\|^2 &= \Big\| \sum_{n=M+1}^{N} c_n e_n \Big\|^2 \\ &= \sum_{n=M+1}^{N} \|c_n e_n\|^2 \\ &= \sum_{n=M+1}^{N} |c_n|^2 = |t_N - t_M|. \end{aligned}$$

Since $\{t_N\}_{N\in\mathbb{N}}$ is Cauchy, we conclude that $\{s_N\}_{N\in\mathbb{N}}$ is Cauchy and hence converges.

(d) By Bessel's inequality and part (c), we know that the series $p = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$ converges, so we just have to show that it is the orthogonal projection of f onto $\overline{\operatorname{span}}(\{e_n\}_{n\in\mathbb{N}})$. Given any k we have by the linearity and continuity of the inner product that

$$\langle f-p, e_k\rangle \;=\; \langle f, e_k\rangle - \sum_{n=1}^{\infty} \langle f, e_n\rangle \langle e_n, e_k\rangle \;=\; \langle f, e_k\rangle - \langle f, e_k\rangle \;=\; 0.$$

By linearity of the inner product, this implies that $f-p \perp \mathrm{span}(\{e_n\}_{n\in\mathbb{N}}$. By continuity of the inner product, this extends to $f-p \perp \overline{\mathrm{span}}(\{e_n\})$. Hence p is indeed the orthogonal projection of f onto $\overline{\mathrm{span}}(\{e_n\}_{n\in\mathbb{N}})$.

4

Now we characterize those sequences in a Hilbert space that are both complete and orthonormal.

Exercise A.97. Let $\{e_n\}_{n\in\mathbb{N}}$ be any orthonormal sequence in a Hilbert space H. Show that the following statements are equivalent.

(a) $\{e_n\}_{n\in\mathbb{N}}$ is complete.

(b) For each $f \in H$ there exist unique scalars $(c_n)_{n\in\mathbb{N}}$ such that $f = \sum_{n=1}^{\infty} c_n e_n$.

(c) For each $f \in H$, $f = \sum_{n=1}^{\infty} \langle f, e_n\rangle\, e_n$.

(d) Plancherel's Equality: For each $f \in H$, $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, e_n\rangle|^2$.

(e) Parseval's Equality: For each $f, g \in H$, $\langle f, g\rangle = \sum_{n=1}^{\infty} \langle f, e_n\rangle \langle e_n, g\rangle$.

5

Proof.

A.97 (a) $\Rightarrow$ (b). If $\{e_n\}$ is complete, then its closed span is all of H, so by Exercise A.96(e) we have $f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$ for every $f \in H$.

(c) $\Rightarrow$ (e). Suppose that (c) holds, and choose any $f, g \in H$. Then

$$\langle f, g \rangle = \left\langle \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, g \right\rangle = \sum_{n=1}^{\infty} \langle \langle f, e_n \rangle e_n, g \rangle = \sum_{n=1}^{\infty} \langle f, e_n \rangle \langle e_n, g \rangle,$$

where we have used Problem A.12 to move the infinite series outside of the inner product.

(d) $\Rightarrow$ (c). Suppose that $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2$ for every $f \in H$. Fix f, and define $s_N = \sum_{n=1}^{N} \langle f, e_n \rangle e_n$. Then, by direct calculation and the by the Pythagorean Theorem,

$$\begin{aligned} \|f - s_N\|^2 &= \|f\|^2 - \langle f, s_N \rangle - \langle s_N, f \rangle + \|s_N\|^2 \\ &= \|f\|^2 - \sum_{n=1}^{N} |\langle f, e_n \rangle|^2 - \sum_{n=1}^{N} |\langle f, e_n \rangle|^2 + \sum_{n=1}^{N} |\langle f, e_n \rangle|^2 \\ &= \|f\|^2 - \sum_{n=1}^{N} |\langle f, e_n \rangle|^2 \to 0 \quad \text{as } N \to \infty. \end{aligned}$$

Hence $f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$.

Thus, *for an orthonormal sequence*, completeness implies the existence of series expansions of vectors. This need not be true for arbitrary complete sequences, even in a Hilbert space (an example is given in Problem A.21).

Definition A.96 (Orthonormal Basis). Let H be a Hilbert space. An orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$ which satisfies the equivalent conditions of Exercise A.95 is called an *orthonormal basis* for H.

Remark A.97. In Section **??**, we will consider *Schauder bases* for Banach spaces. A Schauder basis is a sequence $\{f_n\}_{n\in\mathbb{N}}$ is a Banach space X such that every vector $f \in X$ can be written $f = \sum_{n=1}^{\infty} c_n(f) f_n$ for a unique choice of scalars $c_n(f)$. In this terminology, an *orthonormal* sequence is complete if and only if it is a Schauder basis. However, it is important to emphasize that, in general, an arbitrary complete sequence *need not* be a Schauder basis (see Exercise A.76 or Problem A.21).

We have been considering *countable* orthonormal bases. By Problem A.20, any Hilbert space that has a countable orthonormal basis must be separable. Conversely, Zorn's Lemma can be used to prove that every separable Hilbert space has an orthonormal basis (see Problem A.23). We show explicitly in Theorem 1.112 that $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2[0,1)$.

The results of this section can be extended to uncountable orthonormal sequences in nonseparable Hilbert spaces, but then one should be extremely careful regarding the use of the terminology "basis," as in much of the Banach space literature, the terminology "basis" is reserved for a *countable* sequence that forms a Schauder basis. We summarize without proof the results that hold for uncountable orthonormal sequences.

Theorem A.98. *Let H be a Hilbert space and let I be an index set. If $\{e_i\}_{i\in I}$ is an orthonormal set in H, then the following statements hold.*

(a) *If $f \in H$ then $\langle f, e_i\rangle \neq 0$ for at most countably many $i \in I$.*
(b) *For each $f \in H$, $\sum_{i\in I} |\langle f, e_i\rangle|^2 \leq \|f\|^2$.*
(c) *For each $f \in H$, $\sum_{i\in I} \langle f, e_i\rangle\, e_i$ converges with respect to the net of finite subsets of I.*

Theorem A.99. *Let $\{e_i\}_{i\in I}$ be an orthonormal set in a Hilbert space H. Then the following statements are equivalent.*

(a) *$\{e_i\}_{i\in I}$ is complete.*
(b) *For each $f \in H$, $f = \sum_{i\in I} \langle f, e_i\rangle\, e_i$ with respect to the net of finite subsets of I.*
(c) *For each $f \in H$, $\|f\|^2 = \sum_{i\in I} |\langle f, e_i\rangle|^2$.*

Note that, because of orthogonality, the series that appear in both Theorems A.98 and A.99 contain only countably many nonzero terms. If we restrict to just those countably many nonzero terms, then, in the language of Section A.11, the series converge unconditionally.

Example 5.6 (Fourier Series). Now we give one of the most important examples of an orthonormal basis.

We saw in Exercise 2.2 that if we define $e_n(x) = e^{2\pi i n x}$, then $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal sequence in $L^2[0,1]$ (the space of square-integrable complex-valued functions on the domain $[0,1]$, or we can consider these functions to be 1-periodic functions on the domain $\mathbb{R}$).

It is a fact that $\{e_n\}_{n\in\mathbb{Z}}$ is actually complete in $L^2[0,1]$ and hence is an orthonormal basis for $L^2[0,1]$. The *Fourier coefficients* of $f \in L^2[0,1]$ are

$$\hat{f}(n) = \langle f, e_n\rangle = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx, \qquad n \in \mathbb{Z}.$$

By Theorem 4.17, we have

$$f = \sum_{n\in\mathbb{Z}} \langle f, e_n\rangle\, e_n = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e_n \tag{5.1}$$

(the series converges unconditionally, so it does not matter what ordering of the index set $\mathbb{Z}$ that we use to sum with). Equation 5.1 is called the *Fourier series* of f.

However, note that the series in 5.1 converges *in the norm of the Hilbert space*, i.e., in L^2-norm. That is, the partial sums converge in L^2-norm, i.e.,

$$\lim_{N\to\infty} \left\| f - \sum_{n=-N}^{N} \hat{f}(n)\, e_n \right\|_2 = 0,$$

or,

$$\lim_{N\to\infty} \int_0^1 \left| f(x) - \sum_{n=-N}^{N} \hat{f}(n)\, e^{2\pi i n x} \right|^2 dx = 0.$$

We *cannot* conclude from this that the equality

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}$$

holds *pointwise*. In fact, one of the deepest results of Fourier series is the *Carleson–Hunt Theorem*, which proves the conjecture of Lusin that if $f \in L^p[0,1]$ where $1 < p \le \infty$, then the Fourier series of f converges to f a.e.

Exercise 5.7. Show that the mapping $\mathcal{F}\colon L^2[0,1] \to \ell^2(\mathbb{Z})$ given by $\mathcal{F}(f) = \hat{f} = \{\hat{f}(n)\}_{n\in\mathbb{Z}}$ is exactly the isomorphism constructed by Theorem 5.5 for the case $H_1 = L^2[0,1]$ and $H_2 = \ell^2(\mathbb{Z})$. The operator $\mathcal{F}$ is the *Fourier transform for the circle* (thinking of functions in $L^2[0,1]$ as being 1-periodic, the domain $[0,1]$ is topologically a circle).

Exercise 5.8. Prove the following (easy) special case of the *Riemann–Lebesgue Lemma*: If $f \in L^2[0,1]$ then $\hat{f}(n) \to 0$ as $|n| \to \infty$. The full Riemann–Lebesgue Lemma for the circle states the same conclusion holds if we only assume that $f \in L^1[0,1]$.

Definition 5.9. In honor of Fourier series, if $\{e_n\}_{n\in I}$ is any orthonormal basis of a separable Hilbert space H, then $\{\langle f, e_n\rangle\}_{n\in I}$ is the sequence of *(generalized) Fourier coefficients of* f and $f = \sum_{n\in I}\langle f, e_n\rangle\, e_n$ is the *(generalized) Fourier series of* f.

Exercise 5.10. The Plancherel formula with respect to the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ for $L^2[0,1]$ is

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2.$$

Use the Plancherel formula to derive a formula for π by applying it to the function $f = \chi_{[0,1/2)} - \chi_{[1/2,1)}$.

Additional Problems

A.24. We say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is *ω-independent* if there do not exist scalars c_n, not all zero, such that $\sum_{n=0}^{\infty} c_n x_n = 0$, where the series converges in the norm of H.

(a) Show that every Schauder basis for H is complete and ω-independent.

(b) Let $\alpha,\ \beta \in \mathbb{C}$ be fixed nonzero scalars such that $\left|\frac{\alpha}{\beta}\right| > 1$. Define

$$\begin{aligned} x_0 &= (1,0,0,0,\dots), \\ x_1 &= (\alpha,\beta,0,0,\dots), \\ x_2 &= (0,\alpha,\beta,0,\dots), \end{aligned}$$

etc. Prove that $\{x_k\}_{k\geq 0}$ is complete and finitely linearly independent in ℓ^2, but is not ω-independent and therefore is not a Schauder basis for ℓ^2.

A.25. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in a Hilbert space H. Prove that the following two statements are equivalent.

(a) For each $m \in \mathbb{N}$ we have $f_m \notin \overline{\text{span}}\{f_n\}_{n\neq m}$ (we say that such a sequence is *minimal*).

(b) There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$ in H such that $\langle f_m, g_n\rangle = \delta_{mn}$ for all $m,\ n \in \mathbb{N}$ (we say that sequences $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ satisfying this property are *biorthogonal*).

Show further that, in case these hold, the sequence $\{g_n\}_{n\in\mathbb{N}}$ is unique if and only if $\{f_n\}_{n\in\mathbb{N}}$ is complete.

A.26. Formulate the Gram–Schmidt orthogonalization procedure in an arbitrary inner product space, and use it to show that any infinite-dimensional inner product space contains an infinite orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$.

A.27. Let H be an infinite-dimensional Hilbert space. Show that there exists a series $\sum_{n=1}^{\infty} f_n$ in H that converges unconditionally but not absolutely.

A.28. Use Zorn's Lemma to show that every Hilbert space has a complete orthonormal subset. In particular, every separable Hilbert space therefore has an orthonormal basis.

ZORN'S LEMMA

Partial Order

A partial order $(\mathcal{S}, \leq)$ is a nonempty set together with a relation $\leq$ on $\mathcal{S} \times \mathcal{S}$ which is:

reflexive: $A \leq A \quad \forall A \in \mathcal{S}$

antisymmetric: $A \leq B \ \& \ B \leq A \Rightarrow A = B$

Transitive: $A \leq B \ \& \ B \leq C \Rightarrow A \leq C$.

A partial ordering s.t. every pair $A, B \in \mathcal{S}$ is comparable is a <u>linear</u> or <u>total</u> ordering.

A nonempty subset of $\mathcal{S}$ that is linearly ordered by $\leq$ is a <u>chain</u> in $\mathcal{S}$.

An element $A \in \mathcal{S}$ is <u>maximal</u> in $\mathcal{S}$ if

$$B \in \mathcal{S} \ \& \ B \text{ comparable to } A \Longrightarrow B \leq A.$$

A maximal element need not be comparable to all elements of $\mathcal{S}$, & need not be unique.

An <u>upper bound</u> for a collection $\mathcal{U} \subset \mathcal{S}$ is an element $A \in \mathcal{S}$ s.t. $B \leq A \ \forall B \in \mathcal{U}$.

<u>Zorn's Lemma</u> If $(\mathcal{S}, \leq)$ is partially ordered & every chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$, then $\mathcal{S}$ has a maximal element

Solution sketch.

A.28 Solution sketch. Let $\mathcal{S} = \{S \subseteq H : S \text{ is orthonormal}\}$, i.e., $\mathcal{S}$ is the set of all orthonormal subsets of H. Inclusion of sets is a partial order on $\mathcal{S}$.

Suppose that $\mathcal{C} = \{S_i\}_{i \in I}$ is a chain in H, i.e., for each $i, j \in I$ we have either $S_i \subseteq S_j$ or $S_j \subseteq S_i$. Define $S = \cup_{i \in I} S_i$. Show that S is itself an orthonormal set. Since $S_i \subseteq S$ for every $i \in I$, this says that S is an *upper bound* for the chain $\mathcal{C}$.

Zorn's Lemma says that, given a partially ordered set, if every chain has an upper bound, then the set has a maximal element. Therefore, $\mathcal{S}$ must have a maximal element, i.e., there exists some orthonormal set $T \in \mathcal{S}$ which has the property that if $S \in \mathcal{S}$ and S is comparable to T (i.e., either $S \subseteq T$ or $T \subseteq S$), then we must have $S \subseteq T$.

We claim now that T is a complete orthonormal subset of H. If T was not complete, then $\overline{\text{span}}(T)$ would be a proper subset of H, and hence there would exist some nonzero $f \in \overline{\text{span}}(T)^{\perp}$. By rescaling, we may assume $\|f\| = 1$. But then $T' = T \cup \{f\}$ is orthonormal and $T \subsetneq T'$, contradicting the fact that T is a maximal element of $\mathcal{S}$. Hence T must be complete.