C

~~Functional Analysis and~~ Operator Theory

In this appendix we collect background results on functional analysis and operator theory.

C.1 Linear Operators on Normed Spaces

See Section 5.1 in Folland

In this section we will review the basic properties of linear operators on normed spaces.

Definition C.1 (Notation for Operators). Let X, Y be vector spaces, and let $T\colon X \to Y$ be a function mapping X into Y. We write either $T(f)$ or Tf to denote the image under T of an element $f \in X$.

(a) T is *linear* if $T(\alpha f+\beta g) = \alpha T(f)+\beta T(g)$ for every $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$.

(b) T is *antilinear* if $T(\alpha f+\beta g) = \bar{\alpha} T(f)+\bar{\beta} T(g)$ for $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$.

(c) T is *injective* if $T(f) = T(g)$ implies $f = g$.

(d) The *kernel* or *nullspace* of T is $\ker(T) = \{f \in X : T(f) = 0\}$.

(e) The *range* of T is $\operatorname{range}(T) = \{T(f) : f \in X\}$.

(f) The *rank* of T is the vector space dimension of its range, i.e., $\operatorname{rank}(T) = \dim(\operatorname{range}(T))$. In particular, T is *finite-rank* if $\operatorname{range}(T)$ is finite-dimensional.

(g) T is *surjective* if $\operatorname{range}(T) = Y$.

(h) T is a *bijection* if it is both injective and surjective.

We use either the symbol I or I_X to denote the identity map of a space X onto itself.

A mapping between vector spaces is often referred to as an *operator* or a *transformation*, especially if it is linear. We introduce the following terminology for operators on normed spaces.

Definition C.2 (Operators on Normed Spaces). Let X, Y be normed linear spaces, and let $L\colon X \to Y$ be a linear operator.

(a) L is *bounded* if there exists a finite $K \geq 0$ such that

$$\forall f \in X, \quad \|Lf\| \leq K\,\|f\|.$$

By context, $\|Lf\|$ denotes the norm of Lf in Y, while $\|f\|$ denotes the norm of f in X.

(b) The *operator norm* of L is

$$\|L\| \;=\; \sup_{\|f\|=1} \|Lf\|. \tag{C.1}$$

On those occasions where we need to specify the spaces in question, we will write $\|L\|_{X\to Y}$ for the operator norm of $L\colon X \to Y$.

(c) We set

$$\mathcal{B}(X,Y) \;=\; \{L\colon X \to Y \,:\, L \text{ is bounded and linear}\}.$$

If $X = Y$ then we write $\mathcal{B}(X) = \mathcal{B}(X,X)$.

(f) If $Y = \mathbb{C}$ then we say that L is a *functional.* The set of all bounded linear functionals on X is the *dual space* of X, and is denoted

$$X^* \;=\; \mathcal{B}(X,\mathbb{C}) \;=\; \{L\colon X \to \mathbb{C} \,:\, L \text{ is bounded and linear}\}.$$

Another common notation for the dual space is X'. Note that since the norm on $\mathbb{C}$ is just absolute value, the operator norm of a linear functional $L \in X^* = \mathcal{B}(X,\mathbb{C})$ is

$$\|L\| = \sup_{\|f\|=1} |Lf|.$$

Notation C.3 (Terminology for Unbounded Operators). We mostly will be concerned with bounded operators, but unbounded operators also arise naturally in many circumstances. Very often, unbounded operators are not defined on the entire space X but only on some dense subspace. For example, the differentiation operator $Df = f'$ is certainly not defined on all of $L^p(\mathbb{R})$, but it is common to refer to the "differentiation operator D on $L^p(\mathbb{R})$", with the understanding that D is only defined on some associated dense subspace such as $L^p(\mathbb{R}) \cap C^1(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$. Another common terminology is to write that "$D\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is densely defined," again meaning that the domain of D is a dense subspace of $L^p(\mathbb{R})$ and D maps this domain into $L^p(\mathbb{R})$.

The next exercise establishes some basic properties of linear operators.

Exercise C.4. Let X, Y be normed linear spaces. Let $L\colon X \to Y$ be a linear operator.

(a) L is injective if and only if $\ker L = \{0\}$.

(b) If L is a bijection then the inverse map $L^{-1}: Y \to X$ is also a linear bijection.

(c) L is bounded if and only if $\|L\| < \infty$.

(d) If L is bounded then $\|Lf\| \le \|L\|\,\|f\|$ for every $f \in X$, and $\|L\|$ is the smallest K such that $\|Lf\| \le K\|f\|$ for all $f \in X$.

(e) $\|L\| = \sup_{\|f\|\le 1} \|Lf\| = \sup_{f\neq 0} \dfrac{\|Lf\|}{\|f\|}.$

Example C.5. Consider a linear operator on a finite-dimensional space, say $L: \mathbb{C}^n \to \mathbb{C}^m$. For simplicity, let us impose the Euclidean norm on both $\mathbb{C}^n$ and $\mathbb{C}^m$. If we let $C = \{x \in \mathbb{C}^n : \|x\| = 1\}$ be the unit sphere in $\mathbb{C}^n$, then $L(C) = \{Lx : \|x\| = 1\}$ is a (possibly degenerate) ellipsoid in $\mathbb{R}^m$. The supremum in the definition of the operator norm of L is achieved in this case, and is the length of a semimajor axis of the ellipsoid $L(C)$. Thus, $\|L\|$ is the "maximum distortion" of the unit sphere under L, illustrated for the the case $m = n = 2$ (with real scalars) in Figure C.1.

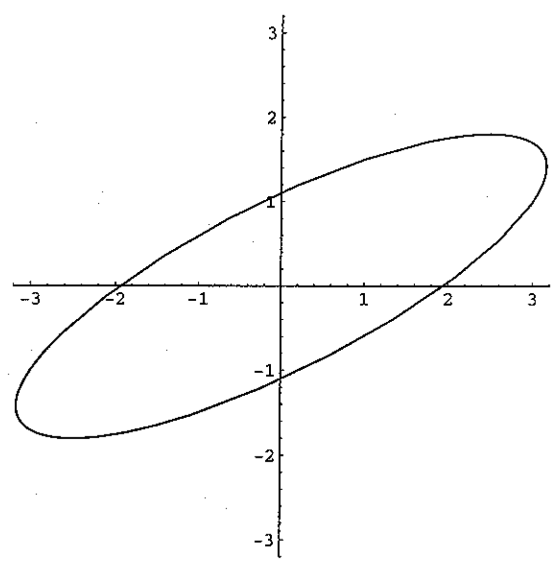

Fig. C.1. Image of the unit circle under a particular linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$. The operator norm $\|L\|$ of L is the length of a semimajor axis of the ellipse.

C.1.1 Equivalence of Boundedness and Continuity

Our first main result of this section shows that continuity is equivalent to boundedness for *linear* operators on normed spaces. Recall that, by

Lemma A.50, if X and Y are normed spaces, then $L\colon X \to Y$ is *continuous at a point* $f \in X$ if $f_n \to f$ in X implies $Lf_n \to Lf$ in Y, and L is *continuous* if it is continuous at every point.

Theorem C.6 (Equivalence of Bounded and Continuous Linear Operators). *If X, Y are normed spaces and $L\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *L is continuous at some $f \in X$.*

(b) *L is continuous at $f = 0$.*

(c) *L is continuous.*

(d) *L is bounded.*

Proof. (c) $\Rightarrow$ (d). Suppose that L is continuous but unbounded. Then we have $\|L\| = \infty$, so there must exist $f_n \in X$ with $\|f_n\| = 1$ such that $\|Lf_n\| \geq n$. Set $g_n = f_n/n$. Then $\|g_n - 0\| = \|g_n\| = \|f_n\|/n \to 0$, so $g_n \to 0$. Since L is continuous and linear, this implies $Lg_n \to L0 = 0$, and therefore $\|Lg_n\| \to \|0\| = 0$. But

$$\|Lg_n\| \;=\; \frac{1}{n}\,\|Lf_n\| \;\geq\; \frac{1}{n}\cdot n \;=\; 1$$

for all n, which is a contradiction. Hence L must be bounded.

(d) $\Rightarrow$ (c). Suppose that L is bounded, so $\|L\| < \infty$. Suppose that $f \in X$ and that $f_n \to f$. Then $\|f_n - f\| \to 0$, so, since L is linear,

$$\|Lf_n - Lf\| \;=\; \|L(f_n - f)\| \;\leq\; \|L\|\,\|f_n - f\| \;\to\; 0,$$

i.e., $Lf_n \to Lf$. Thus L is continuous.

The remaining implications are exercises. □

Thus, if X, Y are normed and $L\colon X \to Y$ is linear, then the terms "continuous" and "bounded" are interchangeable.

C.1.2 Isomorphisms

The notion of a topological isomorphism (or homeomorphism) between arbitrary topological spaces was introduced in Definition A.47. We repeat it here for the case of normed spaces, along with additional terminology for operators that preserve norms.

Definition C.7 (Isometries and Isomorphisms). Let X, Y be normed spaces, and let $L\colon X \to Y$ be linear.

(a) If $L\colon X \to Y$ is a linear bijection such that both L and L^{-1} are continuous, then L is called a *topological isomorphism*, or is said to be *continuously invertible*.

(b) If there exists a topological isomorphism $L: X \to Y$, then we say that X and Y are *topologically isomorphic.*

(c) If $\|Lf\| = \|f\|$ for all $f \in X$ then L is called an *isometry* or is said to be *norm-preserving.*

(d) An isometry $L: X \to Y$ that is a bijection is called an *isometric isomorphism.*

(e) If there exists an isometry $L: X \to Y$ then we say that X and Y are *isometrically isomorphic*, and we write $X \cong Y$ in this case.

On occasion, we will deal with *antilinear isometric isomorphisms*, which are entirely analogous except that the mapping L is antilinear instead of linear.

Remark C.8. The Inverse Mapping Theorem, which is covered in Section ?? below, states that if X and Y are *Banach* spaces and $L: X \to Y$ is a bounded linear bijection, then L^{-1} is automatically bounded and hence L is a topological isomorphism. Thus, when X and Y are Banach spaces, every continuous linear bijection is actually a topological isomorphism.

For the case of Hilbert spaces, we have a special terminology for isometric isomorphisms.

Definition C.9 (Unitary Operator). If H, K are Hilbert spaces and $L: H \to K$ is an isometric isomorphism, then L is called a *unitary operator*, and in this case we say that H and K are *unitarily isomorphic.*

In addition to preserving the norm, an isometry on an inner product space preserves the inner product.

Theorem C.10. *Let H, K be inner product spaces, and let $L: H \to K$ be a linear mapping. Then the following statements are equivalent.*

(a) *L is inner product-preserving, i.e., $\langle Lf, Lg\rangle = \langle f, g\rangle$ for all f, $g \in H$.*

(b) *L is norm-preserving (an isometry), i.e., $\|Lf\| = \|f\|$ for all $f \in H$.*

Proof. (b) $\Rightarrow$ (a). Assume that L is an isometry, and fix f, $g \in H$. Then for any scalar $\lambda \in \mathbb{C}$ we have by the Polar Identity and the fact that L is isometric that

$$\begin{aligned}
\|f\|^2 + 2\,\mathrm{Re}\,\bar{\lambda}\,\langle f, g\rangle + |\lambda|^2\,\|g\|^2 &= \|f + \lambda g\|^2 \\
&= \|Lf + \lambda Lg\|^2 \\
&= \|Lf\|^2 + 2\,\mathrm{Re}\,\bar{\lambda}\,\langle Lf, Lg\rangle + |\lambda|^2\,\|Lg\|^2 \\
&= \|f\|^2 + 2\,\mathrm{Re}\,\bar{\lambda}\,\langle Lf, Lg\rangle + |\lambda|^2\,\|g\|^2.
\end{aligned}$$

Thus $\mathrm{Re}\,\bar{\lambda}\,\langle Lf, Lg\rangle = \mathrm{Re}\,\bar{\lambda}\,\langle f, g\rangle$ for every $\lambda \in \mathbb{C}$. Taking $\lambda = 1$ and $\lambda = i$, this implies that $\langle Lf, Lg\rangle = \langle f, g\rangle$. $\square$

C.1.3 Eigenvalues and Eigenvectors

We recall the definition of the eigenvalues and eigenvectors of an operator that maps a space into itself.

Definition C.11 (Eigenvalues and Eigenvectors). Let X be a normed space and $L\colon X \to X$ a linear operator.

(a) A scalar λ is an *eigenvalue* of L if there exists a nonzero vector $f \in X$ such that $Lf = \lambda f$.

(b) A nonzero vector $f \in X$ is an *eigenvector* of L is there exists a scalar λ such that $Lf = \lambda f$.

Additional Problems

C.1. Let Y be any normed linear space and let $X = \mathbb{C}^d$ under the Euclidean norm. Prove directly that if $L\colon \mathbb{C}^d \to Y$ is linear, then L is bounded.

C.2. Extend Problem C.1 to arbitrary finite-dimensional domains: Show that if X is any finite-dimensional vector space (under any norm) and Y any normed linear space, then every linear function $L\colon X \to Y$ is bounded.

← Hint: All norms on a finite-dim. space are equivalent.

C.3. If X, Y are normed spaces and $L\colon X \to Y$ is continuous, show that $\ker(L)$ is a closed subspace of X.

C.4. (a) Define $L\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $L(x) = (x_2, x_3, \dots)$. Prove that this *left-shift operator* is bounded, linear, surjective, not injective, and is not an isometry. Find $\|L\|$ and all eigenvalues and eigenvectors of L.

(b) Define $R\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $R(x) = (0, x_1, x_2, x_3, \dots)$. Prove that this *right-shift operator* is bounded, linear, injective, not surjective, and is an isometry. Find $\|R\|$ and all eigenvalues and eigenvectors of R.

(c) Compute LR and RL and show that $LR \neq RL$. Contrast this computation with the fact that in finite dimensions, if A, $B\colon \mathbb{C}^n \to \mathbb{C}^n$ are linear maps (hence correspond to multiplication by $n \times n$ matrices), then $AB = I$ implies $BA = I$ and conversely.

C.5. Let X be a Banach space and Y a normed linear space. Suppose that $L\colon X \to Y$ is bounded and linear. Prove that if there exists $c > 0$ such that $\|Lx\| \geq c\|x\|$ for all $x \in X$, then L is injective and range(L) is closed.

C.6. Show that if $L\colon X \to Y$ is a topological isomorphism, then

$$\forall f \in X, \quad \frac{\|f\|}{\|L^{-1}\|} \leq \|Lf\| \leq \|L\|\,\|f\|.$$

C.7. Show that if H, K are separable Hilbert spaces, then H and K are unitarily isomorphic.

C.8. Let A be an $m \times n$ complex matrix, which we view as a linear transformation $A\colon \mathbb{C}^n \to \mathbb{C}^m$. The operator norm of A depends on the choice of norm for $\mathbb{C}^n$ and $\mathbb{C}^m$. Compute an explicit formula for $\|A\|$, in terms of the entries of A, when the norm on $\mathbb{C}^n$ and $\mathbb{C}^m$ is taken to be the ℓ^1 norm. Then do the same for the ℓ^∞ norm. Compare your formulas to the version of Schur's Test given in Theorem C.20.

C.9. The Axiom of Choice implies that every vector space has a basis, which in this volume we refer to as a *Hamel basis*, see Section ??. Use this to show that if X is an infinite-dimensional normed linear space, then there exists a linear functional $\mu\colon X \to \mathbb{C}$ that is unbounded.

Problem

Define
$$\delta : C_b(\mathbb{R}) \longrightarrow \mathbb{C}$$
by
$$\delta(f) = f(0).$$
Show that $\delta \in C_b(\mathbb{R})^*$, and find $\|\delta\|$.

Remark

For reasons that we will see later, we will usually write $\langle f, \delta \rangle$ instead of $\delta(f)$, and adopt a similar notation for all linear functionals.

Problem hints

C.8 Hint: The operator norms are

$$\|A\|_{\ell^1\to\ell^1} = \max_{j=1,\dots,n}\left\{\sum_{i=1}^m |a_{ij}|\right\}$$

and

$$\|A\|_{\ell^\infty\to\ell^\infty} = \max_{i=1,\dots,m}\left\{\sum_{j=1}^n |a_{ij}|\right\}.$$

C.9 Solution. By the Axiom of Choice in the form of Zorn's Lemma, there exists a Hamel basis (vector space basis) $\{f_i\}_{i\in I}$ for X. That is, $\operatorname{span}(\{f_i\}_{i\in I}) = X$ (i.e., the *finite* linear span is all of X), and every finite subset of $\{f_i\}_{i\in I}$ is linearly independent. By dividing each vector by its length, we can assume that $\|f_i\| = 1$ for every $i \in I$. Let $J_0 = \{j_1, j_2, \dots\}$ be any countable subsequence of I. Define $\mu(f_{j_n}) = n$ for $n \in \mathbb{N}$ and $\mu(f_i) = 0$ for $i \in I\backslash J_0$. Then extend μ linearly to all of X: Each nonzero vector $f \in X$ has a unique representation as $f = \sum_{k=1}^N c_k f_{i_k}$ for some $i_1, \dots, i_N \in I$ and nonzero scalars $c_1, \dots, c_N$, so we define $\mu(f) = \sum_{k=1}^N c_k \mu(f_{i_k})$. This μ is a linear functional on X, but since $\|f_{j_n}\| = 1$ yet $|\mu(f_{j_n})| = n$, we have $\|\mu\| = \infty$.

<u>Problem.</u>

Define $\quad \delta' : C_b^1(\mathbb{R}) \longrightarrow \mathbb{R}$

by $\quad \delta'(f) = f'(0).$

(a) Show that $\delta' \in C_b^1(\mathbb{R})^*$ & find $\|\delta'\|$.

(b) Show that if we put the norm $\|\cdot\|_\infty$ on $C_b^1(\mathbb{R})$ instead of $\|f\|_{C_b^1} = \|f\|_\infty + \|f'\|_\infty$, then δ' is not bounded.