C.2.3 Integral Operators

Now we define the important class of *integral operators* for the setting of the real line.

Definition C.16 (Integral Operator). Let k be a fixed measurable function on $\mathbb{R}^2$. Then the *integral operator* L_k *with kernel* k is formally defined by

$$L_k f(x) \;=\; \int k(x,y)\, f(y)\, dy, \tag{C.4}$$

i.e., $L_k f$ is defined whenever this integral makes sense.

An integral operator is a generalization of ordinary matrix multiplication. Let A be an $m \times n$ matrix with entries a_{ij} and let $u \in \mathbb{C}^n$ be given. Then $Au \in \mathbb{C}^m$, and its components are

$$(Au)_i \;=\; \sum_{j=1}^{n} a_{ij}\, u_j, \qquad i = 1, \dots, m.$$

Thus, the function values $k(x,y)$ are analogous to the entries a_{ij} of the matrix A, and the values $L_k f(x)$ are analogous to the entries $(Au)_i$.

Example C.17 (Tensor Product Kernels). The *tensor product* of two functions g, h on $\mathbb{R}$ is the function $g \otimes h$ on $\mathbb{R}^2$ defined by

$$(g \otimes h)(x,y) \;=\; g(x)\, \overline{h(y)}, \qquad x, y \in \mathbb{R}.$$

Sometimes the complex conjugate is omitted in the definition of tensor product, but it will be convenient for our purposes to include it.

An important special case of an integral operator is where the kernel k is a tensor product:

$$k \;=\; g \otimes h.$$

If we assume that g, $h \in L^2(\mathbb{R})$, then for $f \in L^2(\mathbb{R})$ we have for all x for which $g(x)$ is defined that

$$L_k f(x) \;=\; \int g(x)\, \overline{h(y)}\, f(y)\, dy \;=\; \langle f, h \rangle\, g(x).$$

If either $g = 0$ or $h = 0$ then L_k is the zero operator, otherwise the range of L_k is the one-dimensional subspace spanned by g. Thus, L_k is a very "simple" operator in this case: It is a bounded, rank one operator on $L^2(\mathbb{R})$.

Notation C.18. When k is a tensor product, we often identify the operator L_k with the function $g \otimes h$. In other words, we use the same symbols $g \otimes h$ to denote the operator whose rule is

$$(g \otimes h)(f) \;=\; \langle f, h \rangle\, g, \qquad f \in L^2(\mathbb{R}).$$

We can extend this notion of an operator $g \times h$ to arbitrary Hilbert spaces by simply replacing $L^2(\mathbb{R})$ with H on the line above. That is, if $g,\, h \in H$ then $g \otimes h$ is the rank one operator given by

$$(g \otimes h)(f) \;=\; \langle f, h \rangle\, g, \qquad f \in H.$$

Note that if $g = h$ and $\|g\|_2 = 1$, then $g \otimes g$ is the orthogonal projection of H onto the line through g.

Example: The Volterra Operator on $L^2[0,1]$.

Define
$$Lf(x) = \int_0^x f(y)\, dy, \qquad f \in L^2[0,1].$$

This is an integral operator on $L^2[0,1]$ with kernel

$$k(x,y) = \begin{cases} 1, & y \le x \\ 0, & y > x. \end{cases}$$

Since $k \in L^2([0,1]^2)$, L is a Hilbert-Schmidt operator, hence is bounded on $L^2[0,1]$. But we can also easily show this directly:

$$\begin{aligned} \|Lf\|_2^2 &= \int_0^1 |Lf(x)|^2\, dx \\ &= \int_0^1 \left| \int_0^x f(y) \cdot 1\, dy \right|^2 dx \\ &\le \int_0^1 \left(\int_0^x |f(y)|^2 dy \right) \left(\int_0^x 1^2\, dy \right) dx \\ &\le \int_0^1 \|f\|_2^2\, x\, dx \\ &= \frac{1}{2} \|f\|_2^2. \end{aligned}$$

Hence L is bounded, and $\|L\| \le \frac{1}{\sqrt{2}}$.

In general, it is not obvious how to tie properties of the kernel k to properties of the operator L_k. The next two theorems will provide sufficient conditions under which L_k is a bounded operator on $L^2(\mathbb{R})$. First we show that if the kernel is square-integrable, then the corresponding integral operator is a bounded mapping of $L^2(\mathbb{R})$ into itself. This condition on the kernel is particularly interesting, because we will see later that the *Hilbert–Schmidt operators* on $L^2(\mathbb{R})$ are precisely those operators that can be written as integral operators with kernels $k \in L^2(\mathbb{R}^2)$, see Theorem C.85.

Theorem C.19 (Hilbert–Schmidt Integral Operators). *Let $k \in L^2(\mathbb{R}^2)$ be fixed. Then the integral operator L_k given by* (C.4) *defines a bounded mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|L_k\| \le \|k\|_2$.*

Proof. Suppose that $k \in L^2(\mathbb{R}^2)$, and define $k_x(y) = k(x,y)$. Then $k_x \in L^2(\mathbb{R})$ for a.e. x. Hence, if $f \in L^2(\mathbb{R})$, then $L_k f(x) = \langle k_x, \bar{f}\rangle$ exists for almost every x.

To see why $L_k f$ is a measurable and square-integrable function of x, consider first the case where f and k are both nonnegative. Then $k(x,y)\, f(y)$ is a measurable function on $\mathbb{R}^2$, so Tonelli's Theorem tells us that $L_k f(x) = \int k(x,y)\, f(y)\, dy$ is a measurable function of x. We estimate its L^2-norm by applying the Cauchy–Bunyakowski–Schwarz Inequality:

$$\begin{aligned}
\|L_k f\|_2^2 &= \int |L_k f(x)|^2\, dx \\
&= \int \left| \int k(x,y)\, f(y)\, dy \right|^2 dx \\
&\le \int \left(\int |k(x,y)|^2\, dy \right) \left(\int |f(y)|^2\, dy \right) dx \\
&= \int \int |k(x,y)|^2\, dy\, \|f\|_2^2\, dx \\
&= \|k\|_2^2\, \|f\|_2^2 \;<\; \infty.
\end{aligned}$$

Hence $L_k f \in L^2(\mathbb{R})$.

Now suppose that $f \in L^2(\mathbb{R})$ and $k \in L^2(\mathbb{R}^2)$ are arbitrary, and write $f = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$ and $k = (k_1^+ - k_1^-) + i(k_2^+ - k_2^-)$. Each function $L_{k_i^\pm}(f_j^\pm)$ is measurable and belongs to $L^2(\mathbb{R})$. Because they are all square-integrable, we can add them together appropriately to obtain that $L_k f \in L^2(\mathbb{R})$. Now that we know that $L_k f$ is measurable, we can apply the exact same estimates as above to conclude that $\|L_k f\|_2 \le \|k\|_2\, \|f\|_2$. Hence L_k is a bounded mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|L_k\| \le \|k\|_2$. □

There are several distinct results that are often referred to as *Schur's Lemma.* The following result is one of them, although it is usually distinguished from the other Schur's Lemmas by calling it *Schur's Test* (originally formulated in [Sch11]). Here we formulate Schur's test for boundedness of integral operators, but it is instructive to compare this result to Problem C.8, which essentially is Schur's Test for finite matrices.

Theorem C.20 (Schur's Test). *Assume that k is a measurable function on $\mathbb{R}^2$ that satisfies the mixed-norm conditions*

$$\begin{aligned} C_1 &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} \int |k(x,y)|\,dy \;<\; \infty, \\ C_2 &= \operatorname*{ess\,sup}_{y\in\mathbb{R}} \int |k(x,y)|\,dx \;<\; \infty. \end{aligned} \tag{C.5}$$

Then the integral operator L_k given by (C.4) *defines a bounded mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|L_k\| \le (C_1C_2)^{1/2}$.*

Proof. As in the proof of Theorem C.19, measurability of T_kf is most easily shown by showing that T_kf is measurable and square-integrable when f, k are nonnegative, and then extending to the general case. For simplicity, we will omit the details and simply assume that T_kf is measurable for arbitrary $f \in L^2(\mathbb{R})$ and $k \in L^2(\mathbb{R})$. In this case, by applying the Cauchy–Bunyakowski–Schwarz Inequality, we have

$$\begin{aligned} \|L_kf\|_2^2 &= \int |L_kf(x)|^2\,dx \\ &= \int \left| \int k(x,y)\,f(y)\,dy \right|^2 dx \\ &\le \int \left(\int |k(x,y)|^{1/2}\cdot|k(x,y)|^{1/2}\,|f(y)|\,dy \right)^2 dx \\ &\le \int \left(\int |k(x,y)|\,dy \right) \left(\int |k(x,y)|\,|f(y)|^2\,dy \right) dx \\ &\le \int C_1 \int |k(x,y)|\,|f(y)|^2\,dy\,dx \\ &= C_1 \int |f(y)|^2 \int |k(x,y)|\,dx\,dy \\ &\le C_1 \int |f(y)|^2\,C_2\,dy \\ &= C_1C_2\,\|f\|_2^2, \end{aligned}$$

where we have used Tonelli's Theorem to interchange the order of integration. Thus L_k is bounded and $\|L_k\| \le (C_1C_2)^{1/2}$. □

The next exercise shows that the hypotheses of Schur's Test actually yield a much stronger conclusion, namely, boundedness on every $L^p(\mathbb{R})$, not just for $p = 2$.

Exercise C.21. Show that if k satisfies the conditions (C.5), then then L_k is a bounded mapping of $L^p(\mathbb{R})$ into itself for every $1 \le p \le \infty$.

Remark C.22. If we assume only that k is measurable and that $C_2 < \infty$ (with no hypothesis about C_1), then we have that $L_k \colon L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is a bounded mapping. Similarly, if $C_1 < \infty$ then $L_k \colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded. Further, the proofs of these two particular "endpoint cases" are quite simple. Exercise C.21 says that if C_1 and C_2 are both finite, then not only do we have boundedness for the straightforward endpoint cases $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$, but we can also prove the more difficult result of boundedness on $L^p(\mathbb{R})$ for each $1 \le p \le \infty$. This type of extension problem is very common, and indeed there is an entire theory of *interpolation theorems* that deal with similar extension issues, see [BeL76]. One basic interpolation theorem is the *Riesz–Thorin Theorem*, which is discussed in Section 2.3.

C.2.4 Convolution

Convolution is considered in detail in Section 1.3. Here we give another view of convolution by considering it to be a special case of an integral operator. In particular, the convolution of f and g is

$$(f * g)(x) = \int f(y)\, g(x-y)\, dy.$$

Therefore, with g fixed, the mapping $f \mapsto f * g$ is the integral operator L_k whose kernel is $k(x,y) = g(x-y)$.

Exercise C.23. Use Schur's Test to prove the following version of Young's Inequality (Exercise 1.23): If $1 \le p \le \infty$, then

$$\forall\, f \in L^p(\mathbb{R}), \quad \forall\, g \in L^1(\mathbb{R}), \quad \|f * g\|_p \le \|f\|_p\, \|g\|_1.$$

As a consequence, $L^1(\mathbb{R})$ is closed under convolution and is an example of a *Banach algebra* (see Definition C.32).

<u>Problem</u>

Show that $L^1(\mathbb{R})$ is <u>not</u> closed under pointwise products:

$$f, g \in L^1(\mathbb{R}) \not\Rightarrow fg \in L^1(\mathbb{R})$$

Motivation

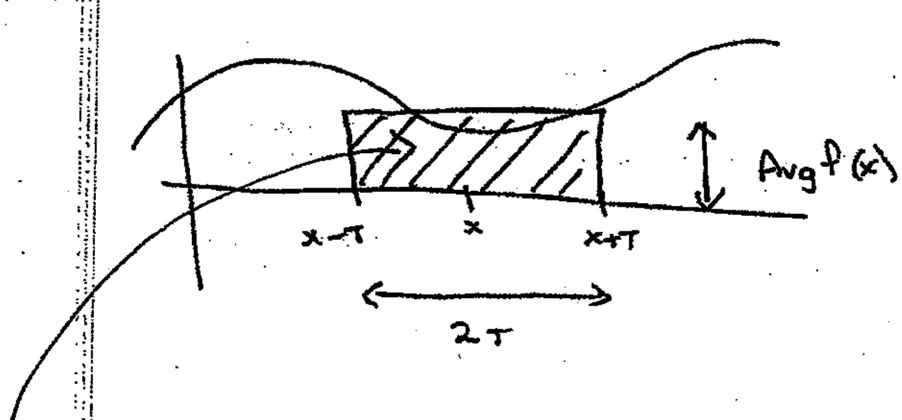
Convolution is a weighted averaging of f.

For example, let $g = \frac{1}{2T}\chi_{[-T,T]}$. Then

$$(f*g)(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

$$= \frac{1}{2T}\int_{x-T}^{x+T} f(y)\, dy$$

$$= \operatorname{Avg} f\,(x)$$



area of the box = $\int_{x-T}^{x+T} f(y)\, dy$ = area under f in $[x-T, x+T]$.

For a general g, $f*g$ can be regarded as a weighted averaging, with g weighting some parts of f more than others.

Since convolution is a type of averaging, it tends to be a "smoothing" process. Define

$$C_b(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous \& bounded}\}.$$

<u>Exercise</u>

$$f \in L^1(\mathbb{R}),\ g \in C_0(\mathbb{R}) \implies f * g \in C_b(\mathbb{R}).$$

Hint: g is uniformly continuous, so

$$\|T_h g - g\|_\infty = \sup_{u \in \mathbb{R}} |g(u-h) - g(u)| \longrightarrow 0 \text{ as } h \to 0.$$

<u>Warning</u>

Some authors write $C(\mathbb{R})$ for what we call $C_b(\mathbb{R})$.
We will take

$$C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is continuous}\}.$$

Thus any polynomial belongs to $C(\mathbb{R})$, but only constant polynomials belong to $C_b(\mathbb{R})$.

Other exercises on convolution

a. If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$f \in L^p(\mathbb{R}),\ g \in L^{p'}(\mathbb{R}) \implies f * g \in C_0(\mathbb{R}).$$

Hint: Let $f_n, g_n \in C_c(\mathbb{R})$ converge to f, g in $L^p, L^{p'}$ norm, respectively. Show that $f_n * g_n \in C_c(\mathbb{R})$ and $f_n * g_n \longrightarrow f*g$ in L^∞-norm.

b. If $1 \leq p \leq \infty$, then

$$f \in L^p(\mathbb{R}),\ g \in C_c(\mathbb{R}) \implies f * g \in C_b(\mathbb{R})$$

c. If $1 \leq p \leq \infty$, then

$$f \in L^p(\mathbb{R}),\ g \in C_c^{(m)}(\mathbb{R}) \implies f * g \in C_b^{(m)}(\mathbb{R})$$

Further, differentiation commutes with convolution:

$$(f*g)^{(j)} = f * g^{(j)}, \qquad j = 0, \dots, m$$

Remark

These results can be used to show that:

$$C_c^{(m)}(\mathbb{R}) \text{ is dense in } L^p(\mathbb{R}), \quad 1 \leq p < \infty$$

$$C_c^{(\infty)}(\mathbb{R}) \text{ is dense in } L^p(\mathbb{R}), \quad 1 \leq p < \infty.$$

Remark

The definition of convolution makes use of the group structure of $\mathbb{R}$. Convolution can be defined more generally, the following is just an illustration.

Exercise

a. Let functions in $L^p[0,1]$ be extended 1-periodically to $\mathbb{R}$, ie, we consider $[0,1]$ to be a group under addition mod 1. Define convolution on this circle group by

$$(f*g)(x) = \int_0^1 f(y)\, g(x-y)\, dy,$$

where the periodicity is used to define $g(x-y)$ when $x-y$ lies outside $[0,1]$. Formulate & prove a version of Young's Inequalities for $L^p[0,1]$.

b. Consider the sequence space $l^p(\mathbb{Z})$, & note that $\mathbb{Z}$ is a group under addition. Define convolution of sequences by

$$(X*y)_n = \sum_{m\in\mathbb{Z}}{}' X_m y_{n-m}.$$

Formulate & prove an analogue of Young's Inequalities for $l^p(\mathbb{Z})$. Show that $l^1(\mathbb{Z})$ is a Banach algebra under convolution, but unlike $L^1(\mathbb{R})$ it does contain an identity, namely $X*\delta = X = \delta*X$ where $\delta = (\ldots, 0, 0, 1, 0, 0, \ldots)$ (1 in the 0^{th} component).

Problem

C.12. Prove the following weighted version of Schur's Test. Assume that k is a measurable function on $\mathbb{R}^2$, and there are measurable functions u, $v \geq 0$ on $\mathbb{R}$ such that

$$\int |k(x,y)|\, v(y)\, dy \;\leq\; C_1\, u(x), \qquad \text{a.e. } x,$$

$$\int |k(x,y)|\, u(x)\, dx \;\leq\; C_2\, v(y), \qquad \text{a.e. } y.$$

Show that the corresponding integral operator L_k defines a bounded mapping of $L^2(\mathbb{R})$ into itself.