C.3 The Space $\mathcal{B}(X, Y)$

We now turn our attention to the space $\mathcal{B}(X,Y)$ of all bounded linear maps from X into Y, which was introduced in Definition C.2.

Exercise C.24. Let V be the vector space containing all functions from X into Y. Show that $\mathcal{B}(X,Y)$ is a subspace of V, and that the operator norm is a norm on $\mathcal{B}(X,Y)$, i.e.,

(a) $0 \leq \|L\| < \infty$ for all $L \in \mathcal{B}(X,Y)$,

(b) $\|L\| = 0$ if and only if $L = 0$ (the zero operator that sends every element of X to the zero vector in Y),

(c) $\|\alpha L\| = |\alpha| \, \|L\|$ for every $L \in \mathcal{B}(X,Y)$ and every $\alpha \in \mathbb{C}$,

(d) $\|L + K\| \leq \|L\| + \|K\|$ for every $L, K \in \mathcal{B}(X,Y)$.

Thus $\mathcal{B}(X,Y)$ is a normed linear space, and we will show in Theorem C.26 that it is a Banach space whenever Y is a Banach space. Before doing this, however, let us note that, in addition to operations of vector addition and scalar multiplication, there is a third operation that we can perform with functions: composition.

Exercise C.25. Prove that the operator norm is *submultiplicative*, i.e., if $A \in \mathcal{B}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$, then $BA \in \mathcal{B}(X,Z)$, and

$$\|BA\| \leq \|B\| \, \|A\|. \tag{C.6}$$

In particular, when $X = Y = Z$, we see that $\mathcal{B}(X)$ is closed under compositions. The space $\mathcal{B}(X)$ is an example of an *Banach algebra* (see Definition C.32).

Theorem C.26. *If X is a normed space and Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space under the operator norm.*

Proof. Assume that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of operators in $\mathcal{B}(X,Y)$ that is Cauchy in operator norm. For any given $f \in X$, we have

$$\|A_m f - A_n f\| \leq \|A_m - A_n\| \, \|f\|,$$

so we conclude that $\{A_n f\}_{n\in\mathbb{N}}$ is a Cauchy sequence of vectors in Y. Since Y is complete, this sequence must converge, say $Af_n \to g \in Y$. Define $Af = g$. This gives us a candidate limit operator A, and we leave as an exercise the task of showing that A defined in this way is linear.

To show that A is bounded, recall that all Cauchy sequences are bounded, so $C = \sup \|A_n\| < \infty$. If $f \in X$, then since $A_n f \to Af$,

$$\|Af\| = \lim_{n\to\infty} \|A_n f\| \leq \sup_{n\in\mathbb{N}} \|A_n f\| \leq \sup_{n\in\mathbb{N}} \|A_n\| \, \|f\| = C \, \|f\|.$$

Therefore A is bounded, with $\|A\| \le C$.

Finally, we must show that $A_n \to A$ in operator norm. Fix any $\varepsilon > 0$. Then there exists an N such that

$$m, n > N \quad \Longrightarrow \quad \|A_m - A_n\| < \frac{\varepsilon}{2}.$$

Choose any $f \in X$ with $\|f\| = 1$. Then since $A_m f \to Af$, there exists an $m > N$ such that

$$\|Af - A_m f\| < \frac{\varepsilon}{2}.$$

Hence for any $n > N$ we have

$$\begin{aligned} \|Af - A_n f\| &\le \|Af - A_m f\| + \|A_m f - A_n f\| \\ &\le \|Af - A_m f\| + \|A_m - A_n\| \, \|f\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Taking the supremum over all unit vectors, we conclude that $\|A - A_n\| \le \varepsilon$ for all $n > N$, so $A_n \to A$. □

Corollary C.27. *If X is any normed space, then its dual space $X^* = \mathcal{B}(X, \mathbb{C})$ is a Banach space.*

The following useful exercise shows that a bounded operator that is defined on a dense subspace of a normed space can be extended to the entire space.

Exercise C.28 (Extension of Bounded Operators). Let Y be a dense subspace of a normed space X, and let Z be a Banach space. Let $L \in \mathcal{B}(Y, Z)$ be given.

(a) Show that there exists a operator $\widetilde{L} \in \mathcal{B}(X, Z)$ whose restriction to Y is L. Prove that $\|\widetilde{L}\| = \|L\|$.

(b) Show that if $L\colon Y \to \text{range}(L)$ is a topological isomorphism, then $\widetilde{L}\colon X \to \overline{\text{range}(L)}$ is a topological isomorphism.

Since $\mathcal{B}(X,Y)$ is a Banach space, we have a natural notion of convergence in $\mathcal{B}(X,Y)$, namely convergence with respect to the operator norm. However, other notions of convergence in $\mathcal{B}(X,Y)$ are often useful, including the following.

Definition C.29 (Strong Operator Topology). Let X, Y be normed spaces, and let L, $L_n \in \mathcal{B}(X,Y)$ be given. We say that L_n *converges to L in the strong operator topology* if $L_n f \to Lf$ for each $f \in H$.

In other words, if we regard $f \in X$ as a point in the space X, then convergence in the strong operator topology is "pointwise convergence" of operators. The next exercise shows that this is a weaker notion than convergence with respect to operator norm.

Exercise C.30. Show that if L_n converges to L in operator norm then it converges in the strong operator topology, but that the converse need not hold.

Additional Problems

C.13. Let H, K be Hilbert spaces, with H separable. Let $L\colon H \to K$ be any bounded, linear operator, and show there exist bounded, finite-rank operators $L_n\colon H \to K$ that converge to L in the strong operator topology.

Exercise hints

C.28 Hints: Fix $f \in X$. Since Y is dense in X, there exist $g_n \in Y$ such that $g_n \to f$. Show that $\{Lg_n\}_{n\in\mathbb{N}}$ is Cauchy in Z, so there exists an $h \in Z$ such that $Lg_n \to h$. Show that $\widetilde{L}f = h$ is well-defined and has the required properties.

C.30 Hint: To construct an example of operators that converge in the strong operator topology but not in operator norm, let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a separable Hilbert space H, and consider the orthogonal projection P_N of H onto $\mathrm{span}\{e_1, \dots, e_N\}$.