C.5 Some Dual Spaces

In this section we consider the dual space of a Hilbert space and the dual space of the Lebesgue space $L^p(E)$. Compare this to the Riesz Representation Theorem given in Appendix D (Theorem D.89), which characterizes the dual space of $C_0(\mathbb{R})$.

C.5.1 The Dual of a Hilbert Space

See Section 5.5 in Folland

If H is a Hilbert space and $h \in H$, then the Cauchy–Bunyakowski–Schwarz Inequality implies that the mapping $\mu_h \colon H \to \mathbb{C}$ given by $\mu_h(f) = \langle f, h\rangle$ is a bounded linear functional on H. The *Riesz Representation Theorem* for Hilbert spaces asserts that every bounded linear functional has this form, and that, consequently, every Hilbert space is "self-dual."

Exercise C.38 (Riesz Representation Theorem). For each $h \in H$, define $\mu_h \colon H \to \mathbb{C}$ by

$$\mu_h(f) = \langle f, h\rangle, \quad f \in H.$$

(a) Show that $\mu_h \in H^*$ for each $h \in H$, and that $\|\mu_h\| = \|h\|$.

(b) For each $\mu \in H^*$, show there exists a unique $h \in H$ such that $\mu = \mu_h$.

(c) Define $T \colon H \to H^*$ by $T(h) = \mu_h$. Prove that T is an antilinear isometric bijection of H onto H^*. In particular,

$$\mu_{\alpha g+\beta h} = \bar{\alpha}\mu_g + \bar{\beta}\mu_h.$$

Proof:

C.38 (a) We have that μ_h is linear, and $|\mu_h(f)| = |\langle f, h\rangle| \le \|f\|\,\|h\|$, so μ_h is bounded and $\|\mu_h\| \le \|h\|$. If $h = 0$, then $\mu_h = 0$ and so $\|\mu_h\| = 0 = \|h\|$. On the other hand, if $h \ne 0$ then $g = h/\|h\|$ is a unit vector, and

$$|\mu_h(g)| \;=\; |\langle g, h\rangle| \;=\; \frac{\langle h, h\rangle}{\|h\|} \;=\; \|h\|,$$

so we have $\|\mu_h\| \ge \|h\|$. Thus $\|\mu_h\| = \|h\|$.

(b) Choose any $\mu \in H^*$. If $\mu = 0$ then $h = 0$ is the required vector, so assume that $\mu \ne 0$ (i.e., μ is not the zero operator). Then μ does not map every vector to zero, so the kernel of μ is a closed subspace of H that is not all of H.

Choose any $g \notin \ker(\mu)$, and write

$$g \;=\; p + e, \quad p \in \ker(\mu), \;\; e \in \ker(\mu)^{\perp}.$$

Note that $\mu(p) = 0$ by definition, and therefore we must have $\mu(e) \ne 0$ (since $\mu(g) \ne 0$). Set $u = e/\mu(e)$, and note that $u \in \ker(\mu)^{\perp}$ and $\mu(u) = 1$.

Given $f \in H$, since μ is linear and $\mu(u) = 1$ we have

$$\mu\big(f - \mu(f)\,u\big) \;=\; \mu(f) - \mu(f)\,\mu(u) \;=\; 0.$$

Therefore $f - \mu(f)\,u \in \ker(\mu)$. However, $u \perp \ker(\mu)$, so

$$0 \;=\; \langle f - \mu(f)\,u, u\rangle \;=\; \langle f, u\rangle - \langle \mu(f)\,u, u\rangle \;=\; \langle f, u\rangle - \mu(f)\,\|u\|^2.$$

Hence,

$$\mu(f) \;=\; \frac{1}{\|u\|^2}\,\langle f, u\rangle \;=\; \Big\langle f, \frac{u}{\|u\|^2}\Big\rangle \;=\; \langle f, h\rangle \;=\; \mu_h(f)$$

where

$$h \;=\; \frac{u}{\|u\|^2}.$$

Thus $L = L_h$, and from part (a), we have that $\|L\| = \|L_h\| = \|h\|$.

It remains only to show that h is unique. Suppose that we also had $\mu = \mu_{h'}$. Then for every $f \in H$ we have

$$\langle f, h - h'\rangle \;=\; \langle f, h\rangle - \langle f, h'\rangle \;=\; \mu_h(f) - \mu_{h'}(f) \;=\; \mu(f) - \mu(f) \;=\; 0.$$

Consequently, $h - h' = 0$.

(c) Parts (a) and (b) show that T is surjective and that T is norm-preserving. Therefore, we just have to show that T is antilinear.

Let $h \in H$ and $c \in \mathbb{C}$ be fixed. We must show that $T(ch) = \bar{c}T(h)$, i.e., that $\mu_{ch} = \bar{c}\mu_h$. This follows immediately from the fact that for each $f \in H$, we have

$$\mu_{ch}(f) \;=\; \langle f, ch\rangle \;=\; \bar{c}\,\langle f, h\rangle \;=\; \bar{c}\,\mu_h(f).$$

The proof that $\mu_{h+k} = T(h+k) = T(h) + T(k) = \mu_h + \mu_k$ is left as an exercise.

Corollary C.39. (a) *If μ is a bounded linear functional on $\ell^2(I)$, then there exists a unique $h = (h_i)_{i\in I} \in \ell^2(I)$ such that*

$$\mu(x) \;=\; \sum_{i\in I} x_i \bar{h}_i \;=\; \langle x, h\rangle, \qquad x = (x_k)_{i\in I} \in \ell^2(I). \tag{C.7}$$

(b) *If μ is a bounded linear functional on $L^2(E)$, then exists a unique $h \in L^2(E)$ such that*

$$\mu(f) \;=\; \int_E f(x)\,\overline{h(x)}\,dx \;=\; \langle f, h\rangle, \qquad f \in L^2(E). \tag{C.8}$$

We usually identify the functional $\mu \in H^*$ with the element $h \in H$ that satisfies $\mu = \mu_h$. However, it is important to note that this identification is *antilinear*. On the other hand, the examples given in equations (C.7) and (C.8) illustrate that this antilinearity is a very natural consequence of the definition of the inner product. For this reason, it is often convenient for us to consider the pairing of a vector f in a normed space X with a linear functional μ on X to be a generalization of the inner product on a Hilbert space, i.e., it is a sesquilinear form that is linear as a function of f but antilinear as a function of μ. We therefore adopt the following notation for denoting the action of a linear functional on a vector. We will use this notation throughout this volume, both for Hilbert spaces and for more general spaces.

Notation C.40 (Notation for Linear Functionals). Let X be a normed linear space. Given a fixed linear functional $\mu\colon X \to \mathbb{C}$, we will have two notations to denote the image of f under μ.

(a) We write

$$\mu(f)$$

to denote the image of f under μ, with the understanding that this notation is linear in both f and μ, i.e.,

$$\mu(\alpha f + \beta g) \;=\; \alpha\mu(f) + \beta\mu(g)$$

and

$$(\alpha\mu + \beta\nu)(f) \;=\; \alpha\mu(f) + \beta\nu(f).$$

(b) We write

$$\langle f, \mu\rangle$$

to denote the image of f under μ, with the understanding that this notation is linear in f but antilinear in μ, i.e.,

$$\langle \alpha f + \beta g, \mu\rangle \;=\; \alpha\langle f, \mu\rangle + \beta\langle g, \mu\rangle$$

while

$$\langle f, \alpha\mu + \beta\nu\rangle \;=\; \bar{\alpha}\langle f, \mu\rangle + \bar{\beta}\langle f, \nu\rangle. \tag{C.9}$$

This will be the preferred notation throughout this volume.

Thus, there is an inherent ambiguity in how we denote linear functionals. If we think of X^* as a vector space, then we would usually consider $\alpha\mu$ to be the functional defined by $(\alpha\mu)(f) = \alpha\mu(f)$. However, in order to follow our preferred notation as given in equation (C.9), we must take $\alpha\mu$ to be the functional defined by $\langle f, \alpha\mu\rangle = \bar{\alpha}\langle f, \mu\rangle$. We always consider the $\langle\cdot,\cdot\rangle$ notation to be a sesquilinear form, linear in the first variable and antilinear in the second.

The reason for this notation is to have a direct extension of the inner product on a Hilbert space. The notation $\mu(f)$, linear in each variable, is not a direct extension of the inner product on H, and this is particularly unpleasant when extending unitary operators beyond the Hilbert space setting. In particular, in order to have a notationally convenient extension of the Parseval formula $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$ for the Fourier transform (Section 2.1), it is important that the form $\langle\cdot,\cdot\rangle$ be sesquilinear. The Parseval formula is one of the major motivations for our choice of notation, and we will shortly discuss some other motivations.

Problem: Explicitly characterize $(\mathbb{C}^n)^*$

Notes

1. All linear maps with a finite-dimensional domain are continuous.

2. Any linear map $T: \mathbb{C}^n \to \mathbb{C}^m$ is given by an $m \times n$ matrix.

3. Identifying linear maps on $\mathbb{C}^n$ with matrices, show that if we take $\mathbb{C}^n$ to be the set of column vectors of length n, then $(\mathbb{C}^n)^*$ is the set of row vectors of length n.
   Thus we have a natural isomorphism

$$(\mathbb{C}^n)^* \cong \mathbb{C}^n$$

Q. Is this isomorphism linear or antilinear?
Does it extend the usual dot product on $\mathbb{C}^n$?

A. It depends on how we do it.

Given $y \in \mathbb{C}^n$, define $\mu_y \in (\mathbb{C}^n)^*$ by

$$\mu_y(x) = x \cdot y = x_1\overline{y_1} + \cdots + x_n\overline{y_n}$$

This matches the usual dot product, but

$y \mapsto \mu_y$ is an antilinear map of $\mathbb{C}^n$ onto $(\mathbb{C}^n)^*$.

On the other hand, given $y \in \mathbb{C}^n$, define $\nu_y \in (\mathbb{C}^n)^*$ by

$$\nu_y(x) = [y_1 \cdots y_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n.$$

This does not match the inner product on $\mathbb{C}^n$, but

$y \mapsto \nu_y$ is a linear map of $\mathbb{C}^n$ onto $(\mathbb{C}^n)^*$