C.5.2 The Dual of $L^p(E)$

The fact that the dual space of Hilbert space $L^2(E)$ is (antilinearly) isomorphic to $L^2(E)$ has a generalization to $L^p(E)$ for $1 \le p \le \infty$. By Hölder's Inequality, if $g \in L^{p'}(E)$, then $\langle f, \mu_g \rangle = \int_E f(x)\, \overline{g(x)}\, dx$ defines a bounded linear functional on $L^p(E)$, and the following exercise shows that the operator norm of μ_g equals the $L^{p'}$ norm of g.

Exercise C.41. Let E be a Lebesgue measurable subset of $\mathbb{R}$, and fix $1 \le p \le \infty$. For each $g \in L^{p'}(E)$, define $\mu_g \colon L^p(E) \to \mathbb{C}$ by

$$\langle f, \mu_g \rangle = \int_E f(x)\, \overline{g(x)}\, dx, \qquad f \in L^p(E). \tag{C.10}$$

Show that $\mu_g \in L^p(E)^*$ and $\|\mu_g\| = \|g\|_{p'}$.

Note

By definition of operator norm,

$$\|\mu_g\| = \sup_{\|f\|_p = 1} |\langle f, \mu_g \rangle|$$

Equivalently, $\|\mu_g\|$ is the smallest number such that

$$|\langle f, \mu_g \rangle| \le \|f\|_p \, \|\mu_g\|, \quad \text{all } f \in L^p(E).$$

Proof:

C.41 Hölder's Inequality implies that $\|\mu_g\| \le \|g\|_{p'}$.

(a) Fix $1 < p < \infty$. Choose $g \in L^{p'}(E)$. If $g = 0$ a.e., then $\mu_g = 0$ and $\|\mu_g\| = \|g\|_{p'}$. Therefore, we can assume that g is not the zero vector.

Let $|\alpha(x)| = 1$ satisfy $\alpha(x)\,\overline{g(x)} = |g(x)|$, and define

$$f(x) \;=\; \frac{\alpha(x)\,|g(x)|^{p'-1}}{\|g\|_{p'}^{p'-1}}.$$

Then since

$$p' \;=\; \frac{p}{p-1} \qquad \text{and} \qquad p \;=\; \frac{p'}{p'-1},$$

we have $(p'-1)\,p = p'$, so

$$\|f\|_p^p \;=\; \int_E \left(\frac{|g(x)|^{p'-1}}{\|g\|_{p'}^{p'-1}}\right)^p dx \;=\; \int_E \frac{|g(x)|^{p'}}{\|g\|_{p'}^{p'}}\,dx \;=\; 1.$$

Also,

$$|\langle f, \mu_g\rangle| \;=\; \int_E f(x)\,\overline{g(x)}\,dx \;=\; \int_E \frac{\alpha(x)\,|g(x)|^{p'-1}}{\|g\|_{p'}^{p'-1}}\,\overline{g(x)}\,dx$$

$$=\; \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'-1}} \;=\; \|g\|_{p'}.$$

This shows that $\|\mu_g\| \ge \|g\|_{p'}$. Hence we have that $\|\mu_g\| = \|g\|_{p'}$.

(b) Now consider the case $p = 1$. Fix any nonzero $g \in L^\infty(\mathbb{R})$. Choose $\varepsilon > 0$. Then there exists a set $A \subseteq E$ with $0 < |A| < \infty$ such that $|g(x)| \ge \|g\|_\infty - \varepsilon$ for a.e. $x \in A$. Let $f = \frac{1}{|A|}\chi_A$. Then $\|f\|_1 = 1$, and

$$\langle f, g\rangle \;=\; \frac{1}{|A|}\int_A |g(x)|\,dx \;\ge\; \frac{1}{|A|}\int_A \big(\|g\|_\infty - \varepsilon\big)\,dx \;=\; \|g\|_\infty - \varepsilon.$$

Hence $\|\mu_g\| \ge \|g\|_\infty - \varepsilon$, and since ε is arbitrary we conclude that $\|\mu_g\| = \|g\|_\infty$.

(c) Finally, consider $p = \infty$. Fix any nonzero $g \in L^1(\mathbb{R})$. As before, let $|\alpha(x)| = 1$ satisfy $\alpha(x)\,\overline{g(x)} = |g(x)|$, and define $f(x) = \alpha(x)$ for all x. Then $\|f\|_\infty = 1$, and

$$\langle f, g\rangle \;=\; \int_E \alpha(x)\,\overline{g(x)}\,dx \;=\; \int_E |g(x)|\,dx \;=\; \|g\|_1.$$

This shows that $\|\mu_g\| \ge \|g\|_1$, so we have that $\|\mu_g\| = \|g\|_1$.

Although we will not prove it, the next theorem states that if $1 \le p < \infty$ then every bounded linear functional on $L^p(E)$ has the form μ_g for some $g \in L^{p'}(\mathrm{e})$. Consequently, $L^p(E)^*$ and $L^{p'}(E)$ are (antilinearly) isomorphic. However, for $p = 1$, although we obtain an embedding of $L^1(E)$ into $L^\infty(E)^*$, this mapping is not surjective. The proof of Theorem C.42 relies on the Radon–Nikodym Theorem (see Theorem D.59).

Theorem C.42 (Dual Space of $L^p(E)$). *Let E be a Lebesgue measurable subset of $\mathbb{R}$, and fix $1 \le p < \infty$. For each $g \in L^{p'}(E)$, define μ_g as in equation* (C.10). *Then the mapping $T\colon L^{p'}(E) \to L^p(E)^*$ defined by $T(g) = \mu_g$ is an antilinear isometric isomorphism of $L^{p'}(E)$ onto $L^p(E)^*$.*

There is an analogous result for the ℓ^p spaces, and generalizations to $L^p(X)$ for arbitrary measure spaces X, see [Fol99].

REAL ANALYSIS LECTURE NOTES:

THE DUAL OF L^p

CHRISTOPHER HEIL

Notation: Integrals with unspecified measures are taken to be Lebesgue integrals. That is, we write

$$\int f = \int f(x)\,dx.$$

We begin with a kind of converse to the fact that each element of $L^{p'}(E)$ defines a bounded linear functional on $L^p(E)$.

Theorem 1. Let E be a Lebesgue measurable subset of $\mathbb{R}$, and fix $1 \le p \le \infty$. Let S be the set of simple functions on E that vanish outside a set of finite measure, i.e.,

$$S = \Big\{\phi \colon E \to \mathbb{C} : \phi \text{ is simple and } |\{\phi \ne 0\}| < \infty\Big\}.$$

Suppose that:

(a) $g \colon E \to \mathbb{C}$ is Lebesgue measurable,

(b) $\phi\, g \in L^1(E)$ for each $\phi \in S$, and

(c) $M_g = \sup\left\{\left|\int \phi\, g\right| : \phi \in S,\ \|\phi\|_p = 1\right\} < \infty.$

Then $g \in L^{p'}(E)$ and $\|g\|_{p'} = M_g$.

Proof. By Hölder's Inequality, we have

$$\left|\int \phi\, g\right| \le \|\phi\|_p\, \|g\|_{p'},$$

so we automatically have $M_g \le \|g\|_{p'}$.

Let $\alpha(x)$ be the complex scalar of unit modulus such that

$$|g(x)| = \alpha(x)\, g(x), \qquad x \in E.$$

Step 1. Note that if $\phi \in S$, then $\phi \in L^p(E)$ and by definition of M_g we have

$$\left|\int \phi\, g\right| \le M_g\, \|\phi\|_p.$$

We will extend this formula to bounded measurable functions f on E that vanish outside a set of finite measure.

Suppose that f is bounded and measurable, and that f is nonzero only on a set A of finite measure. Then $\chi_A \in S$, so $\chi_A\, g \in L^1(E)$. Since f is bounded and zero outside of A, we therefore have $fg \in L^1(E)$.

By our standard approximation theorems, we can find simple functions ϕ_k such that $\phi_k \to f$ pointwise a.e. and $|\phi_k| \le |f|$. Then each ϕ_k belongs to S, and we have $\phi_k\, g \to fg$ pointwise a.e. and $|\phi_k\, g| \le |fg| \in L^1(E)$. Hence, by the Lebesgue Dominated Convergence Theorem,

$$\left|\int fg\right| \;=\; \lim_{k\to\infty}\left|\int \phi_k\, g\right| \;\le\; \lim_{k\to\infty} M_g\,\|\phi_k\|_p \;\le\; M_g\,\|f\|_p. \tag{1}$$

Step 2. Suppose now that $1 < p < \infty$, so we also have $1 < p' < \infty$. Let ϕ_k be simple functions on E such that $0 \le \phi_k \nearrow |g|^{p'}$. If necessary, by replacing ϕ_k with $\phi_k \cdot \chi_{E\cap[-k,k]}$ we may assume that each ϕ_k vanishes outside a set of finite measure. Since each ϕ_k is nonnegative, we can define

$$g_k \;=\; \alpha\cdot\phi_k^{1/p}.$$

Motivation:

$$|g_k| \;=\; \phi_k^{1/p} \;=\; \phi_k^{1-\frac{1}{p'}} \;\approx\; \big(|g|^{p'}\big)^{1-\frac{1}{p'}} \;=\; |g|^{p'-1},$$

and therefore

$$|g_k\, g| \;\approx\; |g|^{p'}.$$

More precisely, since $|g_k| = \phi_k^{1/p}$, we have

$$\|g_k\|_p \;=\; \left(\int |g_k|^p\right)^{1/p} \;=\; \left|\int \phi_k\right|^{1/p} \;=\; \|\phi_k\|_1^{1/p}$$

and

$$\phi_k \;=\; \phi_k^{1/p}\,\phi_k^{1/p'} \;\le\; \phi_k^{1/p}\,|g| \;=\; \phi_k^{1/p}\,\alpha\, g \;=\; g_k\, g.$$

Further, each g_k is bounded and vanishes outside a set of finite measure, so by equation (1) we have

$$\|\phi_k\|_1^{1/p}\,\|\phi_k\|_1^{1/p'} \;=\; \|\phi_k\|_1 \;=\; \int \phi_k \;\le\; \int g_k\, g \;\le\; M_g\,\|g_k\|_p \;=\; M_g\,\|\phi_k\|_1^{1/p}.$$

If $g = 0$ then there is nothing to prove. Otherwise, we will have $\phi_k \ne 0$ for all large enough k, and therefore we can divide through in the equation above to obtain

$$\|\phi_k\|_1^{1/p'} \;\le\; M_g.$$

By Fatou's Lemma, we therefore have

$$\|g\|_{p'}^{p'} \;=\; \int |g|^{p'} \;=\; \int \liminf_{k\to\infty} |\phi_k| \;\le\; \liminf_{k\to\infty} \int |\phi_k| \;=\; \liminf_{k\to\infty} \|\phi_k\|_1 \;\le\; M_g^{p'}.$$

Thus $g \in L^{p'}(E)$ and $\|g\|_{p'} \le M_g$.

Step 3. Suppose that $p = 1$. Fix $\varepsilon > 0$, and define

$$A \;=\; \big\{|g| \ge M_g + \varepsilon\big\}.$$

If $|A| > 0$, choose $B \subseteq A$ with $0 < |B| < \infty$. Define

$$f \;=\; \alpha \cdot \chi_B.$$

Then f is bounded and vanishes outside a set of finite measure, so by equation (1) we have

$$\left|\int fg\right| \;\leq\; M_g \, \|f\|_1 \;=\; |B|.$$

However,

$$\int fg \;=\; \int_B \alpha g \;=\; \int_B |g| \;\geq\; (M_g + \varepsilon)\,|B|.$$

Therefore

$$(M_g + \varepsilon)\,|B| \;\leq\; M_g\,|B|,$$

which is a contradiction. Therefore we must have $|A| = 0$. Hence $|g| \leq M_g + \varepsilon$ a.e. Since this is true for every $\varepsilon > 0$, we conclude that $\|g\|_\infty \leq M_g < \infty$.

Step 4. Suppose $p = \infty$, so we have $p' = 1$. Set

$$g_k \;=\; \alpha\,\chi_{E\cap[-k,k]}.$$

Then g_k is bounded an vanishes outside a set of finite measure, so we have

$$\int_{E\cap[-k,k]} |g| \;=\; \int g_k\, g \;\leq\; M_g\, \|g_k\|_\infty \;=\; M_g.$$

Since this is true for every k, we have

$$\|g\|_1 \;=\; \int_E |g| \;\leq\; M_g \;<\; \infty.$$

Hence $g \in L^1(E)$. □

Remark 2. This theorem extends to arbitrary positive measure spaces (X, Σ, μ) if we assume either that the set $S_g = \{g \neq 0\}$ is σ-finite, or that the measure μ is semifinite. See Theorem 6.14 in Folland for details.

Now we can characterize the dual of $L^p(E)$ for finite p.

Theorem 3 (Dual Space of $L^p(E)$). Let E be a Lebesgue measurable subset of $\mathbb{R}$, and fix $1 \leq p < \infty$. For each $g \in L^{p'}(E)$, define μ_g by

$$\langle f, \mu_g\rangle \;=\; \int_E f(x)\,\overline{g(x)}\,dx, \qquad f \in L^p(E). \tag{2}$$

Then the mapping $T\colon L^{p'}(E) \to L^p(E)^*$ defined by $T(g) = \mu_g$ is an antilinear isometric isomorphism of $L^{p'}(E)$ onto $L^p(E)^*$.

Proof. We already know that T is an antilinear isometric map of $L^{p'}(E)$ into $L^p(E)^*$. Therefore T is injective, and hence we only need to prove that T is surjective.

Case 1: $|E| < \infty$. Suppose that $\mu \in L^p(E)^*$, i.e., $\mu\colon L^p(E) \to \mathbb{C}$ is bounded and linear. Define

$$\nu(A) = \langle \chi_A, \mu \rangle, \qquad \text{measurable } A \subseteq E.$$

Our immediate goal is to show that ν is a complex measure on $(E, \mathcal{L}_E)$, where $\mathcal{L}_E$ is the σ-algebra of Lebesgue measurable subsets of E.

First, since E has finite measure, we have $\chi_A \in L^p(E)$ for each measurable $A \subseteq E$. Therefore $\nu(A)$ is a well-defined complex scalar for each $A \subseteq E$, and also $\nu(\emptyset) = \langle 0, \mu \rangle = 0$.

Second, suppose that $A_1, A_2, \dots$ are disjoint measurable subsets of E, and let $A = \cup A_j$. Then we have pointwise that

$$\chi_A(x) = \sum_{j=1}^{\infty} \chi_{A_j}(x), \qquad x \in E,$$

since for each x the series on the right has at most one nonzero term.

Exercise: Show that the series $\chi_A = \sum_{j=1}^{\infty} \chi_{A_j}$ converges in L^p-norm (note that we use the fact that $|E| < \infty$ here).

Consequently, using both the linearity *and* the continuity of μ, we have

$$\nu(A) = \langle \chi_A, \mu \rangle = \left\langle \sum_{j=1}^{\infty} \chi_{A_j}, \mu \right\rangle = \sum_{j=1}^{\infty} \langle \chi_{A_j}, \mu \rangle = \sum_{j=1}^{\infty} \nu(A_j).$$

Therefore ν is countably additive, and hence is a complex measure on E.

Moreover, if $A \subseteq E$ satisfies $|A| = 0$, then $\chi_A = 0$ a.e., so we have

$$\nu(A) = \langle \chi_A, \mu \rangle = \langle 0, \mu \rangle = 0.$$

Hence $\nu \ll dx$.

The Radon–Nikodym Theorem therefore implies that there exists a $g \in L^1(E)$ such that $d\nu = \bar{g}\, dx$. We will show that $g \in L^{p'}(E)$ and that $\mu = \mu_g = T(g)$, which will show that T is surjective.

Now, the symbols $d\nu = \bar{g}\, dx$ mean that

$$\nu(A) = \int_A \overline{g(x)}\, dx, \qquad \text{measurable } A \subseteq E.$$

Consequently, for each measurable $A \subseteq E$ we have

$$\langle \chi_A, \mu \rangle = \nu(A) = \int \chi_A \, d\nu = \int \chi_A \, \bar{g}.$$

By linearity, given any simple function ϕ on E, we therefore have

$$\langle \phi, \mu \rangle = \int \phi \, \bar{g}.$$

Since $|E| < \infty$, any simple function belongs to $L^p(E)$. Since μ is bounded on $L^p(E)$, we therefore have that

$$\left|\int \phi\, g\right| \;=\; |\langle \phi, \mu\rangle| \;\le\; \|\mu\|\, \|\phi\|_p.$$

It therefore follows from Theorem 1 that $g \in L^{p'}(E)$. Hence μ_g is a bounded linear functional on $L^p(E)$, i.e., $\mu_g \in L^p(E)^*$. For any simple function, we have

$$\langle \phi, \mu_g\rangle \;=\; \int \phi\, \bar{g} \;=\; \langle \phi, \mu\rangle.$$

Thus μ and μ_g agree on the simple functions. Since these operators are continuouus and since the simple functions are dense in $L^p(E)$, we conclude that $\mu = \mu_g$. This finishes the proof for this case.

Case 2: $|E| = \infty$. Set $E_k = E \cap [-k, k]$ (the important fact here being that the real line under Lebesgue measure is σ-finite). Let μ_k be the restriction of μ to $L^p(E_k)$. Then $\mu_k \in L^p(E_k)^*$, so it follows from Case 1 that there exists a function $g_k \in L^{p'}(E_k)$ such that $\mu_k = \mu_{g_k}$. By taking functions to be zero outside of E_k, we can consider $L^p(E_k)$ to be a subspace of $L^p(E)$. Therefore, we have

$$\langle f, \mu\rangle \;=\; \langle f, \mu_k\rangle \;=\; \int_{E_k} f\, \overline{g_k}, \qquad f \in L^p(E_k).$$

Furthermore,

$$\|g_k\|_{p'} \;=\; \|\mu_k\| \;\le\; \|\mu\|,$$

where μ_k is the operator norm of μ restricted to $L^p(E_k)$, and $\|\mu\|$ is the operator norm of μ on $L^p(E)$.

Now, if f is any function in $L^p(E_k)$, then we have $f \in L^p(E_{k+1})$ as well, and since f is zero outside of E_k, we have

$$\int_{E_k} f\, \overline{g_k} \;=\; \langle f, \mu\rangle \;=\; \int_{E_{k+1}} f\, \overline{g_{k+1}} \;=\; \int_{E_k} f\, \overline{g_{k+1}}.$$

Consequently, we must have $g_k = g_{k+1}$ for almost every $x \in E_k$. Hence we can define a function g on E by setting $g(x) = g_k(x)$ if $x \in E_k$. Then g is measurable, and if $1 < p < \infty$ then we have by Fatou's Lemma that

$$\|g\|_{p'} \;\le\; \liminf_{k\to\infty} \|g_k\|_{p'} \;\le\; \|\mu\| \;<\; \infty.$$

If $p = 1$ then $p' = \infty$ and we have

$$\|g\|_\infty \;=\; \sup_k \|g_k\|_\infty \;\le\; \|\mu\| \;<\; \infty.$$

In any case, we conclude that $g \in L^{p'}(E)$.

Finally, if $f \in L^p(E)$ then $f\chi_{E_k} \in L^p(E_k)$, so

$$\langle f\chi_{E_k}, \mu\rangle \;=\; \int_{E_k} f\, \overline{g_k} \;=\; \int_{E_k} f\, \bar{g}.$$

However, $f\chi_{E_k} \to f$ in L^p-norm and μ is continuous, so

$$\langle f\chi_{E_k}, \mu\rangle \to \langle f, \mu\rangle.$$

On the other hand, $fg \in L^1(E)$ by Hölder's Inequality, so by the Lebesgue Dominated Convergence Theorem, we have

$$\int_{E_k} f\,\bar{g} \to \int_E f\,\bar{g}.$$

Hence

$$\langle f, \mu\rangle = \int_E f\,\bar{g} = \langle f, \mu_g\rangle.$$

This is true for all $f \in L^p(E)$, so we conclude that $\mu = \mu_g$. □

Remark 4. This theorem extends to arbitrary positive measure spaces (X, Σ, μ). For $1 < p < \infty$ we can even prove this without any restrictions on μ, i.e., even if μ is not σ-finite. However, for $p = 1$ we must assume that μ is σ-finite. See Theorem 6.15 in Folland for details.

C.5.3 The Relation between $L^{p'}(E)$ and $L^p(E)^*$

We have chosen to consider the relation between $L^{p'}(E)$ and $L^p(E)^*$ in a way that most directly generalizes the inner product on a Hilbert space and the characterization of the dual space of a Hilbert space given by the Riesz Representation Theorem. Under our choice, we write the action of $\mu \in L^p(E)^*$ on $f \in L^p(E)$ as $\langle f, \mu \rangle$, and regard this as a sesquilinear form, linear in f but antilinear in μ. With this notation, the following statements hold.

(a) $L^p(E)$, $L^{p'}(E)$, and $L^p(E)^*$ are linear spaces.

(b) $L^p(E)^*$ is the space of bounded linear functionals on $L^p(E)$.

(c) $T\colon L^{p'}(E) \to L^p(E)^*$ given by $T(g) = \mu_g$ is an isometric isomorphism, but is antilinear.

To illustrate one advantage of this approach, consider the special case $p = 2$. Since $L^2(E)$ is both a Hilbert space and a particular L^p space, we have introduced two different uses of the notation $\langle \cdot, \cdot \rangle$ with regard to $L^2(E)$. On the one hand, $\langle f, g \rangle$ denotes the inner product of $f,\ g \in L^2(E)$, while, on the other hand, $\langle f, \mu \rangle$ denotes the action of $\mu \in L^2(E)^*$ on $f \in L^2(E)$. Fortunately, $\langle f, g \rangle = \langle f, \mu_g \rangle$, so our linear functional notation is not in conflict with our inner product notation. This notationally simplifies certain calculations. For example, if $A\colon L^2(E) \to L^2(E)$ is unitary then we have for $f,\ g \in L^2(E)$ that $\langle f, g \rangle = \langle Af, Ag \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on H, and also $\langle f, \mu_g \rangle = \langle Af, \mu_{Ag} \rangle$. However, we do have to accept that our identification of g with μ_g is antilinear rather than linear.

There are several alternative approaches, each with their own advantages and disadvantages. Let us discuss two of these.

A second choice is to base our notation on the usual convention that if ν is a linear functional, then the notation $\nu(f)$ is linear in both f and ν. If we follow this convention, then we will associate a function $g \in L^{p'}(E)$ with the functional $\nu_g\colon L^p(\mathbb{R}) \to \mathbb{C}$ defined by

$$\nu_g(f) \;=\; \int f(x)\, g(x)\, dx.$$

With this notation, we have the following facts.

(a) $L^p(E)$, $L^{p'}(E)$, and $L^p(E)^*$ are linear spaces.

(b) $L^p(E)^*$ is the space of bounded linear functionals on $L^p(E)$.

(c) $U\colon L^{p'}(E) \to L^p(E)^*$ given by $U(g) = \nu_g$ is an isometric isomorphism, and is linear.

This is a natural choice except for the fact that the notation $\nu(f)$ is not an extension of the inner product on $L^2(E)$. Specifically, although we identify $g \in L^2(\mathbb{R})$ with $\nu_g \in L^2(E)^*$, the inner product $\langle f, g \rangle$ does not coincide with $\nu_g(f)$. Hence for $p = 2$, if $A\colon L^2(E) \to L^2(E)$ is unitary, then although we have $\langle f, g \rangle = \langle Af, Ag \rangle$, we do *not* have $\nu_g(f) = \nu_{Ag}(Af)$. Another consequence is that if $L\colon L^2(E) \to L^2(E)$ is linear, then the adjoint L^* of L defined by the requirement that $\langle Lf, g\rangle = \langle f, L^*g\rangle$ is different than the adjoint defined by the requirement that $\nu_g(Lf) = \nu_{L^*g}(f)$ (adjoints are considered in Section C.6). Essentially, we end up with different notions for concepts on $L^2(E)$ depending on whether we regard $L^2(E)$ as a Hilbert space under the inner product, or a member of the class of Banach spaces $L^p(E)$ with the identification between $L^p(E)^*$ and $L^{p'}(E)$ given by U. The isomorphism U between $L^2(E)$ and $L^2(E)^*$ is different than the one given by the Riesz Representation Theorem (Exercise C.38).

A third possibility is to let the functionals on $L^p(E)$ be antilinear functionals instead of linear. For example, we can associate $g \in L^{p'}(E)$ with the functional $\rho_g\colon L^p(\mathbb{R}) \to \mathbb{C}$ given by

$$[f, \rho_g] = \int \overline{f(x)}\, g(x)\, dx.$$

Then the the dual space is a space of antilinear functionals, i.e., the dual space is

$$L^p(E)^{\neg} = \{\rho\colon L^p(E) \to \mathbb{C} : \rho \text{ is bounded and antilinear}\}.$$

In this case, we have the following facts.

(a) $L^p(E)$, $L^{p'}(E)$, and $L^p(E)^{\neg}$ are linear spaces.

(b) $L^p(E)^{\neg}$ is the linear space whose elements are the bounded antilinear functionals on $L^p(E)$.

(c) $V\colon L^{p'}(E) \to L^p(E)^{\neg}$ given by $V(g) = \rho_g$ is an isometric isomorphism, and is linear.

While V is linear, we again have a disagreement between our functional notation and the inner product on $L^2(E)$.

Despite the fact that our discussion of notation has been quite lengthy, in the end the difference between these choices comes down to nothing more than convenience — each choice makes certain formulas "pretty" and others "unpleasant." As our main concern is the use of these notations in harmonic analysis, our choice is motivated by the formulas of harmonic analysis, and in particular the Parseval formula for the Fourier transform. We choose a notation that directly generalizes the inner product, and consequently obtain the simplest notational representation for generalizing the Fourier transform to distributions and measures (see Chapters 4 and 5).