X^{**} and Reflexivity

We know that if X is a normed space, then X^* is a Banach space. Hence $X^{**} = (X^*)^*$, $X^{***} = ((X^*)^*)^*$, etc., are all Banach spaces. If X is a Hilbert space, then $X \cong X^*$ (and in fact, the converse is also true, $X \cong X^*$ implies X is a Hilbert space).

In this section we will explore the relation between X & X^{**} for general normed spaces.

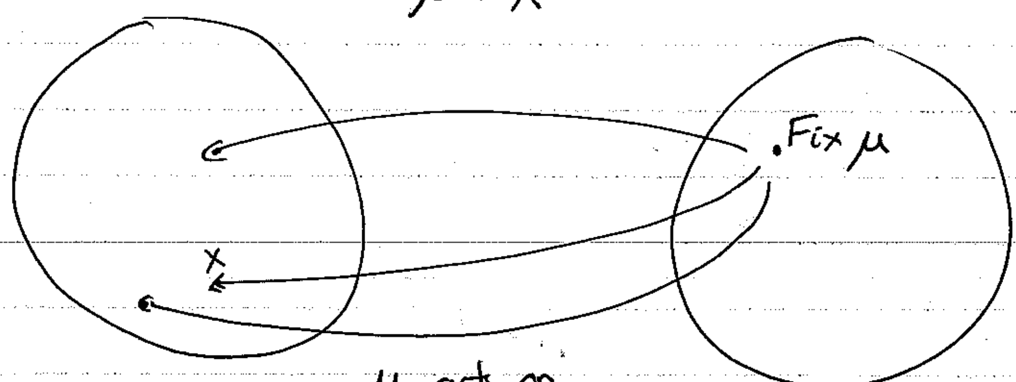
Motivation

Choose $\mu \in X^*$. Then, by definition, $x \longmapsto \langle x, \mu \rangle$ is a bounded linear functional on X. But we could also fix $x \in X$ and consider the mapping $\mu \longmapsto \langle x, \mu \rangle$. This is a mapping from X^* to $\mathbb{F}$, and under our notational conventions, it is an antilinear functional on X^*. This is one

$\mathbb{F}$ = our scalar field, either $\mathbb{R}$ or $\mathbb{C}$

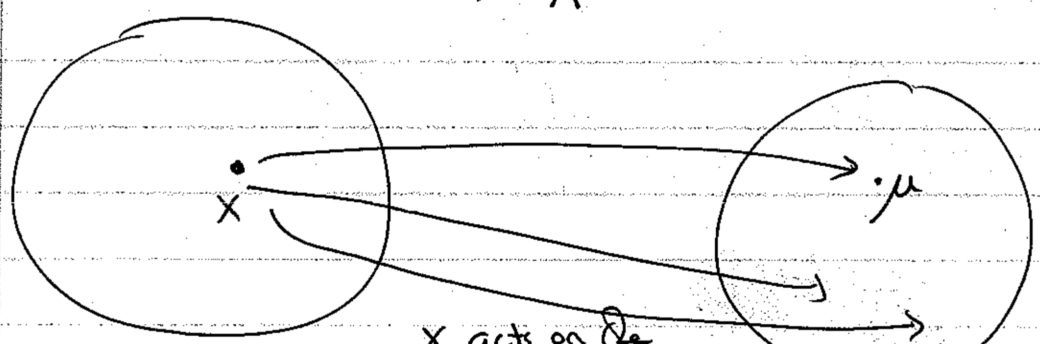
Thus each $x \in X$ determines an $\hat{x} \in X^{**}$. Not only that, but the mapping $x \longmapsto \hat{x}$ is an isometry, so there is a natural isomorphic embedding of X onto X^{**}. If this map is onto, then we will say that X is <u>reflexive</u>.

$\mu \in X^*$



X | μ acts on all the elements $x \in X$, $x \longmapsto \mu(x)$ | X^*

$x \in X$



X | x acts on the elements of X^* $\mu \longmapsto \mu(x)$ | X^*

place where our choice of notational convention is less convenient, so for this section let us return to the standard functional notation

$$\mu(x) \quad \text{instead of} \quad \langle x, \mu \rangle.$$

We regard this notation as being linear in <u>both</u> x and μ.

So, again, we have that if $\mu \in X^*$ is fixed, then $x \longmapsto \mu(x)$ is a bounded linear functional on X, but also if we fix $x \in X$ then $\mu \longmapsto \mu(x)$ is a linear functional on X^*. Since this functional is determined by x, we call it $\hat{x}$. That is:

$$x \in X \implies \begin{aligned} \hat{x} : X^* &\longrightarrow \mathbb{F} \\ \mu &\longmapsto \mu(x) \end{aligned}$$

is a linear functional on X^*.

Further, we'll see that $\hat{x}$ is bounded, so $\hat{x} \in X^{**}$.

Theorem

Let X be a normed linear space.

a. Given $x \in X$, define

$$\hat{x} : X^* \longrightarrow \mathbb{F}$$
$$\mu \longmapsto \mu(x)$$

Then $\hat{x} \in X^{**}$, i.e., $\hat{x}$ is a bounded linear functional on X^*.

b. We have $\|\hat{x}\| = \|x\|$ for each $x \in X$.

$$\underset{\substack{\uparrow \\ \text{operator} \\ \text{norm of } \hat{x}}}{\|\hat{x}\|} = \underset{\substack{\uparrow \\ \text{norm of } x \in X}}{\|x\|}$$

Consequently,

$$T : X \longrightarrow X^{**}$$
$$x \longmapsto \hat{x}$$

is a linear isometry of X into X^{**}.

Proof

a. Fix $x \in X$. Exercise: $\hat{x}$ is linear. For $\mu \in X^*$,

$$|\hat{x}(\mu)| = |\mu(x)| \le \underbrace{\|\mu\|}_{\substack{\uparrow \\ \text{This is } \|\mu\|_{X^*},\ \text{op. norm of } \mu.}} \|x\|.$$

Therefore

$$\|\hat{x}\|_{X^{**}} = \sup_{\|\mu\|_{X^*}=1} |\hat{x}(\mu)|$$

$$\leq \sup_{\|\mu\|_{X^*}=1} \|\mu\|_{X^*} \|x\|$$

$$= \|x\|.$$

Thus $\hat{x}$ is bounded, so $\hat{x} \in X^{**}$.

b. Exercise: T is linear.

So, it remains only to show that $\|\hat{x}\|_{X^{**}} = \|x\|$.

This follows from (a corollary to) Hahn-Banach:

$$\|\hat{x}\|_{X^{**}} = \sup_{\|\mu\|_{X^*}=1} |\hat{x}(\mu)|$$

$$= \sup_{\|\mu\|_{X^*}=1} |\mu(x)|$$

$$= \|x\| \quad \text{by Hahn-Banach.}$$

Thus T is an isometry. ▨

Definition

If the map T of the preceding theorem is surjective, then we say that X is reflexive.

Thus, if X is reflexive, then $T: X \to X^{**}$ is an isometric isomorphism, so $X \cong X^{**}$.

Note: Since X^{**} is a Banach space, only Banach spaces can be reflexive.

Remark: $X \cong X^{**}$ does not imply that X is reflexive — in order to be reflexive, the isomorphism must be given by the "natural map" T.

Notation

We call T the natural map of X into X^{**}. As usual, we identify $x \in X$ with its image $\hat{x} \in X^{**}$, and therefore write

$$X \subseteq X^{**}$$

in the sense of identification under the natural map. If X is reflexive, then $X = X^{**}$, again in the sense of identification under the natural map.

Exercise

Fix $1 < p < \infty$. Show that $\ell^p(\mathbb{N})$ and $L^p(E)$, $E \subseteq \mathbb{R}$ Lebesgue measurable, are reflexive.

Note: It is not quite enough to say that we have shown that $L^p(E)^* \cong L^{p'}(E)$, & therefore

$$L^p(E)^{**} \cong L^{p'}(E)^* \cong L^{p''}(E) = L^p(E).$$

This is true, but to be reflexive the isomorphism $L^p(E) \longrightarrow L^p(E)^{**}$ must be given by the natural map.

Exercise

Recall that $c_0^* \cong \ell^1$ & $(\ell^1)^* \cong \ell^\infty$, so $c_0^{**} \cong \ell^\infty$. Show that the natural map $T: c_0 \longrightarrow \ell^\infty$ is the inclusion map, i.e., $Tx = x$ for $x \in c_0$. Conclude that c_0 is not reflexive (why?).

(Do this exercise after we have covered Hahn-Banach.)

<u>Exercise</u> (See Conway, p. 92-93)

Let M be a closed subspace of a Banach space X

Let $\rho_X : X \to X^{**}$ & $\rho_M : M \to M^{**}$ be the natural maps.

Let $i : M \to X$ be the inclusion map, $i(x) = x \ \forall x \in M$.

Show $\exists$ isometry $\phi : M^{**} \to X^{**}$ s.t.

$$\rho_X \circ i = \phi \circ \rho_M.$$

$$\begin{tikzcd} X \arrow[r, "\rho_X"] & X^{**} \\ M \arrow[u, "i"] \arrow[r, "\rho_M"'] & M^{**} \arrow[u, dashed, "\exists \phi"'] \end{tikzcd}$$

Prove further that $\phi(M^{**}) = (M^{\perp})^{\perp}$.

<u>Exercise</u>

Use the preceding exercise to show that if X is reflexive, then any closed subspace of X is reflexive.