FUNCTIONAL ANALYSIS LECTURE NOTES:

PRODUCT SPACES

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We will define the product or direct sum of a countable collection of normed spaces. An analogous definition holds for the case of a finite collection of spaces.

Definition 0.1. Let $\{X_i\}_{i\in\mathbb{N}}$ be a countable family of normed linear spaces, and let $\|\cdot\|_i$ denote the norm on X_i. Define

$$\prod_{i=1}^{\infty} X_i \;=\; \bigl\{f=(f_1,f_2,\dots) : f_i \in X_i\bigr\}.$$

For $1 \le p < \infty$, define

$$\bigoplus\nolimits_p X_i \;=\; \Biggl\{ f \in \prod_{i=1}^{\infty} X_k \;:\; \|f\|_p \;=\; \biggl(\sum_{i=1}^{\infty} \|f_i\|_i^p\biggr)^{1/p} < \infty \Biggr\}.$$

For $p=\infty$, define

$$\bigoplus\nolimits_\infty X_i \;=\; \Biggl\{ f \in \prod_{i=1}^{\infty} X_k \;:\; \|f\|_\infty \;=\; \sup_i \|f_i\|_i < \infty \Biggr\}.$$

Exercise 0.2. Let $\{X_i\}_{i\in\mathbb{N}}$ be a countable family of normed spaces and fix $1 \le p \le \infty$. Let $X = \bigoplus_p X_i$. Prove the following.

(a) X is a normed space.

(b) For each i, the projection $P_i \colon X \to X_i$ given by $P_i(f_1,f_2,\dots) = f_i$ is continuous, and $\|P_i\| = 1$.

(c) X is a Banach space if and only if each X_i is a Banach space.

Exercise 0.3. Let $X_1,\dots,X_n$ be finitely many normed spaces. Prove that the spaces $\oplus_p X_i$ are equal for $1 \le p \le \infty$, and that all the norms $\|\cdot\|_p$ are equivalent.

In light of the preceding exercise, when dealing with finitely many spaces $X_1,\dots,X_n$, we often denote $\oplus_p X_i$ by $X_1 \times \cdots \times X_n$, without reference to the choice of norm $\|\cdot\|_p$.

Exercise 0.4. Let $X_1,\dots,X_n$ be finitely many normed spaces. Given $y_k = (y_k(1),\dots,y_k(n))$ and $y = (y(1),\dots,y(n)) \in X_1 \times \cdots \times X_n$, show that

$$y_k \to y \text{ in } X_1 \times \cdots \times X_n \qquad \Longleftrightarrow \qquad y_k(j) \to y(j) \text{ in } X_j,\; j=1,\dots,n.$$

Exercise 0.5. Let X and Y be normed vector spaces. Define $T\colon \mathcal{B}(X,Y) \times X \to Y$ by $T(A,f) = Af$.

(a) Prove that T is continuous if and only if $A_n \to A$ and $f_n \to f$ implies $A_n f_n \to Af$.

(b) Prove that T is continuous. Conclude that $T \in \mathcal{B}(\mathcal{B}(X,Y) \times X,\, Y)$.