

The Hahn-Banach Theorem: Preparation

Notation Reminder: Linear Functionals

If X is a vector space, we'll use letters x, y, z or f, g, h to denote elements of X , and μ, ν, ρ etc. to denote linear functionals on X .

We use a generalized inner-product notation to denote the action of a functional μ on a vector x . That is, if $\mu: X \rightarrow \mathbb{R}$ or \mathbb{C} , then we write $\langle x, \mu \rangle$ instead of $\mu(x)$ for the value of μ at x . Since μ is linear,

$$\langle \alpha f + \beta g, \mu \rangle = \alpha \langle f, \mu \rangle + \beta \langle g, \mu \rangle.$$

However, as a function of μ this notation is anti-linear:

$$\langle f, \alpha \mu + \beta \nu \rangle = \bar{\alpha} \langle f, \mu \rangle + \bar{\beta} \langle f, \nu \rangle,$$

just like an inner product on a Hilbert space.

Chapter 3: Linear Functionals & the Hahn-Banach Theorem

In this chapter it will be important on occasion to distinguish between results that hold for the real field versus the complex field. We will use the symbol \mathbb{F} to denote a generic choice of \mathbb{R} or \mathbb{C} .

Linear Functionals

We will develop some basic facts relating properties of a linear functional to properties of its kernel. Some of the ~~more~~ results only concern linearity & hence are valid on general vector spaces, while others concern boundedness & unboundedness & hence hold on normed spaces.

Definition

A subspace M of a vector space X is a hyperplane in X if $\dim(X/M) = 1$, i.e., M has codimension 1.

Motivation: Hilbert Spaces

Suppose that $\mu: H \rightarrow \mathbb{F}$ is a bounded linear functional on a Hilbert space H . Then by the Riesz Representation Theorem, $\exists g \in H$ s.t.

$$\langle f, \mu \rangle = \langle f, g \rangle \quad \forall f \in H.$$

Hence $\ker(\mu) = g^\perp$ is a closed subspace of H .

Consequently,

$$H / \ker(\mu) \cong \ker(\mu)^\perp = \text{span}\{g\}$$

is 1-dimensional, so $\ker(\mu)$ is a hyperplane in H .

Remark

In a general normed space, we have no analogue of orthogonal complements. In a Hilbert space, if M is a closed subspace then $H/M \cong M^\perp$, & usually we just work with M & M^\perp . We have the fundamental decomposition

$$H = M \oplus M^\perp.$$

This is lost in the non-Hilbert space setting, but

often, by working with X/M instead of the nonexistent M^\perp , we can obtain at least some analogues of results that hold for Hilbert spaces.

The following result will be an important tool. This is the vector space analogue of a result known in abstract algebra as the (First) Isomorphism Theorem or the (First) Homomorphism Theorem. In the group setting of abstract algebra, an isomorphism is a bijective homomorphism. In the vector space setting, an isomorphism is a linear bijection.

Exercise: The Isomorphism Theorem

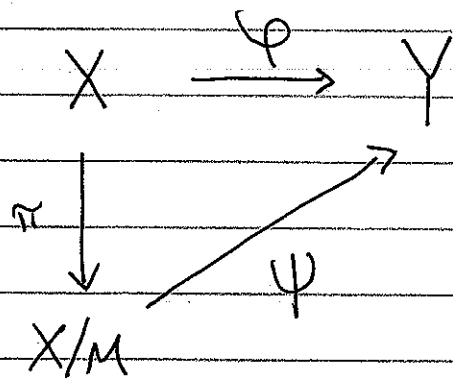
Let X, Y be vector spaces. Let $\varphi: X \rightarrow Y$ be a linear surjection. Let $M = \ker(\varphi)$, and let $\pi: X \rightarrow X/M$ be the canonical projection. Prove that

$$\begin{aligned} \psi: X/M &\longrightarrow Y \\ f+M &\longmapsto \varphi(f) \end{aligned}$$

is a well-defined linear bijection, and that

$$\varphi = \psi \circ \pi,$$

i.e., the following diagram commutes:



Example/Exercise

Define $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}$
 $(a, b) \mapsto b.$

Then $M = \ker(\psi)$ is the x-axis in \mathbb{R}^2 . Hence

$$\mathbb{R}^2/M = \{M + (0, b) : b \in \mathbb{R}\}$$

and $\psi: M + (0, b) \mapsto b$ is a linear bijection

of \mathbb{R}^2/M onto \mathbb{R} . Show that ψ is the bijection

given by the Isomorphism Theorem, i.e., ~~$\psi \circ \pi$~~ $\psi \circ \pi = \psi.$

Remark

So far, we have only considered \mathbb{R} vector space properties of X & Y . If X, Y are normed, we can add \mathbb{R} into consideration in \mathbb{R} formulation of \mathbb{R} Isomorphism Theorem.

Theorem The Isomorphism Theorem for Normed Spaces

Let X, Y be normed spaces. Let $\varphi: X \rightarrow Y$ be a bounded linear surjection, $M = \ker(\varphi)$, and let $\psi: X/M \rightarrow Y$ be \mathbb{R} linear surjection given by \mathbb{R} Isomorphism Theorem for vector spaces. Then

$$\|\psi\| = \|\varphi\| \quad (\text{equal operator norms})$$

Proof:

Letting $\pi: X \rightarrow X/M$ be \mathbb{R} canonical projection, we have $\varphi = \psi \circ \pi$. Therefore

$$\|\varphi\| = \|\psi \circ \pi\| \leq \|\psi\| \|\pi\| = \|\psi\|.$$

To establish \mathbb{R} converse inequality, recall that

$$\|\Psi\| = \sup_{\|x+M\|_{X/M}=1} \|\Psi(x+M)\|$$

Hence we can find $x_n \in X$ such that $\|x_n+M\|_{X/M} = 1$ and

$$\|\Psi(x_n+M)\| \nearrow \|\Psi\|.$$

Now,

$$1 = \|x_n+M\|_{X/M} = \inf_{m \in M} \|x_n+m\|,$$

so we can find $m_n \in M$ such that

$$\|x_n+m_n\| \rightarrow 1.$$

Hence

$$\|\Psi\| \geq \frac{\|\Psi(x_n+m_n)\|}{\|x_n+m_n\|}$$

$$= \frac{\|\Psi(x_n)\|}{\|x_n+m_n\|} \quad \text{since } M = \ker(\Psi)$$

$$= \frac{\|\Psi(x_n+M)\|}{\|x_n+m_n\|}$$

$$\rightarrow \|\Psi\|.$$

This gives the required inequality. \blacksquare

Now we can show that every hyperplane is the kernel of some (not necessarily continuous) linear functional.

Proposition

Let M be a subspace of a normed space X . Then TFAE:

a. M is a hyperplane.

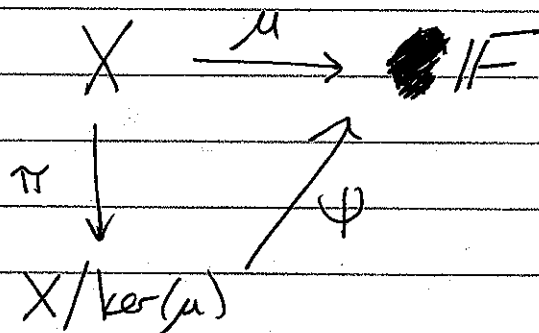
b. $M = \ker(\mu)$ for some nonzero linear functional $\mu: X \rightarrow \mathbb{F}$.

Proof

b \Rightarrow a. Suppose that $M = \ker(\mu)$, where μ is a nonzero

linear functional on X . Then ~~by the~~ by the Isomorphism

Theorem, \exists linear bijection $\psi: X/\ker(\mu) \rightarrow \mathbb{F}$.



Since \mathbb{F} is 1-dimensional, we conclude that $X/\ker(\mu)$

is also 1-dimensional.

$a \Rightarrow b$. Assume that M is a hyperplane in X . Then

X/M is 1-dimensional, so \exists linear bijection $\Psi: X/M \rightarrow \mathbb{F}$

Set $\varphi = \Psi \circ \pi$. Then $\varphi: X \rightarrow \mathbb{F}$ is linear.

Exercise: Show $\ker(\varphi) = M$

Finally, since $\dim(X/M) = 1$, we have

$M \subsetneq X$ (why?). Hence $\ker(\varphi) \neq X$, so φ is

not the zero functional. \blacksquare

The next result is a bit surprising at first glance, but less so when we consider that kernels of linear functionals are hyperplanes, hence are "almost" the entire space except for "one dimension."

Proposition

Let X be a vector space. If $\mu, \nu: X \rightarrow F$ are nonzero linear functionals, then

$$\ker(\mu) = \ker(\nu) \iff \mu = c\nu \text{ for some nonzero } c \in F.$$

Proof:

\Rightarrow Suppose that $\ker(\mu) = \ker(\nu)$. Since $\mu \neq 0$, $\exists f \in X$

s.t. $\langle f, \mu \rangle \neq 0$, and by rescaling we may assume

$\langle f, \mu \rangle = 1$. Then $f \notin \ker(\mu) = \ker(\nu)$, so $\langle f, \nu \rangle \neq 0$.

Given any $g \in X$, we have

$$\langle g - \langle g, \mu \rangle f, \mu \rangle = \langle g, \mu \rangle - \langle g, \mu \rangle \langle f, \mu \rangle = 0.$$

Therefore $g - \langle g, \mu \rangle f \in \ker(\mu) = \ker(\nu)$. Hence

$$\langle g, \nu \rangle - \langle g, \mu \rangle \langle f, \nu \rangle = \langle g - \langle g, \mu \rangle f, \nu \rangle = 0.$$

Hence

$$\langle g, \nu \rangle = \langle \langle f, \nu \rangle g, \mu \rangle \quad \forall g \in X$$

so

$$\nu = \langle f, \nu \rangle \mu,$$

and, as we said, $\langle f, \nu \rangle \neq 0$. \square

Now we turn from results valid for general vector spaces to the setting of normed spaces, where ~~we~~ we can consider continuity of functionals and closedness of subspaces.

Proposition

If M is a hyperplane in a normed linear space X , then either M is closed or M is dense in X .

Proof:

We are given that X/M is 1-dimensional. Let $\pi: X \rightarrow X/M$ be the canonical projection. The closure \bar{M} of M is a subspace of X . Since π is linear, $\pi(\bar{M})$ must be a subspace of X/M . But X/M is 1-dimensional, so there are only two possibilities.

First, we could have $\pi(\bar{M}) = \{0+M\}$.

In this case, $\bar{M} \subseteq \ker(\pi) = M$, which implies

$\bar{M} = M$, so M is closed.

Second, we could have $\pi(\bar{M}) = X/M$. In this case $\bar{M} = X$ (why?), so M is dense in X . \blacksquare

Finally, we obtain a useful characterization of bounded linear functionals on a normed space.

Proposition

Let $\mu: X \rightarrow \mathbb{F}$ be a linear functional on a normed space X .

Then:

μ is continuous $\iff \ker(\mu)$ is closed.

Proof

\Rightarrow Exercise.

\Leftarrow Suppose that $M = \ker(\mu)$ is closed. We know that

M is a hyperplane, so $\dim(X/M) = 1$. Let

$\pi: X \rightarrow X/M$ be the canonical projection. Since π

~~is a linear map~~ $\ker(\pi) = M$ is closed, a previous

result implies that π is continuous.

Now, by the Isomorphism Theorem, \exists linear

bijection $\psi: X/M \rightarrow \mathbb{F}$, although the


Isomorphism Theorem doesn't tell us whether ψ is

Continuous.

$$\begin{array}{ccc} X & \xrightarrow{\mu} & \mathbb{F} \\ \pi \downarrow & \nearrow \psi & \\ X/M & & \end{array}$$

However, any linear map from a finite-dimensional space to a normed space is continuous (previous exercise), so in

fact ψ must be continuous. Therefore

$\mu = \psi \circ \pi$ is continuous as well. 

Alternatively, the Isomorphism Theorem for

normed spaces tells us that $\|\psi\| = \|\mu\|$,

so μ is bounded.