The Hahn-Banach Theorem: Preparation

<u>Notation Reminder</u>: Linear Functionals

If X is a vector space, we'll use letters x, y, z or f, g, h to denote elements of X, and μ, ν, ρ etc. to denote linear functionals on X.

We use a generalized inner-product notation to denote the action of a functional μ on a vector X. That is, if $\mu: X \longrightarrow \mathbb{R}$ or $\mathbb{C}$, then we write $\langle f, \mu \rangle$ instead of $\mu(f)$ for the value of μ at f. Since μ is linear,

$$\langle \alpha f + \beta g, \mu \rangle = \alpha \langle f, \mu \rangle + \beta \langle g, \mu \rangle.$$

However, as a function of μ this notation is anti-linear:

$$\langle f, \alpha\mu + \beta\nu \rangle = \bar{\alpha} \langle f, \mu \rangle + \bar{\beta} \langle f, \nu \rangle,$$

just like an inner product on a Hilbert space.

Chapter 3: Linear Functionals & the Hahn-Banach Theorem

In this chapter it will be important on occasion to distinguish between results that hold for the real field versus the complex field. We will use the symbol $\mathbb{F}$ to denote a generic choice of $\mathbb{R}$ or $\mathbb{C}$.

Linear Functionals

We will develop some basic facts relating properties of a linear functional to properties of its kernel. Some of the results only concern linearity & hence are valid on general vector spaces, while others concern boundedness & unboundedness & hence hold on normed spaces.

Definition

A subspace M of a vector space X is a hyperplane in X if $\dim(X/M) = 1$, i.e., M has codimension 1.

Motivation: Hilbert Spaces

Suppose that $\mu: H \longrightarrow \mathbb{F}$ is a bounded linear functional on a Hilbert space H. Then by the Riesz Representation Theorem, $\exists g \in H$ s.t.

$$\langle f, \mu \rangle = \langle f, g \rangle \qquad \forall f \in H.$$

Hence $\ker(\mu) = g^{\perp}$ is a closed subspace of H.

Consequently,

$$H / \ker(\mu) \cong \ker(\mu)^{\perp} = \operatorname{span}\{g\}$$

is 1-dimensional, so $\ker(\mu)$ is a hyperplane in H.

Remark

In a general normed space, we have no analogue of orthogonal complements. In a Hilbert space, if M is a closed subspace then $H/M \cong M^{\perp}$, & usually we just work with M & $M^{\perp}$. We have the fundamental decomposition

$$H = M \oplus M^{\perp}.$$

This is lost in the non-Hilbert space setting, but often, by working with X/M instead of the nonexistent $M^\perp$, we can obtain at least some analogues of results that hold for Hilbert spaces.

The following result will be an important tool. This is the vector space analogue of a result known in abstract algebra as the (First) Isomorphism Theorem or the (First) Homomorphism Theorem. In the group setting of abstract algebra, an isomorphism is a bijective homomorphism. In the vector space setting, an isomorphism is a linear bijection.

Exercise: The Isomorphism Theorem

Let X, Y be vector spaces. Let $\varphi: X \longrightarrow Y$ be a linear surjection. Let $M = \ker(\varphi)$, and let $\pi: X \longrightarrow X/M$ be the canonical projection. Prove that

$$\begin{aligned} \psi: X/M &\longrightarrow Y \\ f+M &\longmapsto \varphi(f) \end{aligned}$$

is a well-defined linear bijection, and that

$$\varphi = \psi \circ \pi,$$

i.e., the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\pi"'] & Y \\ X/M \arrow[ur, "\psi"'] & \end{tikzcd}$$

Example / Exercise

Define $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$
$$(a,b) \longmapsto b.$$

Then $M = \ker(\varphi)$ is the x-axis in $\mathbb{R}^2$. Hence

$$\mathbb{R}^2/M = \{M + (0,b) : b \in \mathbb{R}\}$$

and $\psi : M + (0,b) \longmapsto b$ is a linear bijection of $\mathbb{R}^2/M$ onto $\mathbb{R}$. Show that ψ is the bijection given by the Isomorphism Theorem, i.e., $\varphi \circ \pi = \psi$.

Remark

So far, we have only considered the vector space properties of X & Y. If X, Y are normed, we can add this into consideration in the formulation of the Isomorphism Theorem.

Theorem The Isomorphism Theorem for Normed Spaces

Let X, Y be normed spaces. Let $\varphi : X \to Y$ be a bounded linear surjection, $M = \ker(\varphi)$, and let $\psi : X/M \to Y$ be the linear surjection given by the Isomorphism Theorem for vector spaces. Then

$$\|\psi\| = \|\varphi\| \quad \text{(equal operator norms).}$$

Proof:

Letting $\pi : X \to X/M$ be the canonical projection, we have $\varphi = \psi \circ \pi$. Therefore

$$\|\varphi\| = \|\psi \circ \pi\| \leq \|\psi\| \, \|\pi\| = \|\psi\|.$$

To establish the converse inequality, recall that

$$\|\Psi\| = \sup_{\|x+M\|_{X/M}=1} \|\Psi(x+M)\|.$$

Hence we can find $x_n \in X$ such that $\|x_n+M\|_{X/M} = 1$ and

$$\|\Psi(x_n+M)\| \nearrow \|\Psi\|.$$

Now,

$$1 = \|x_n+M\|_{X/M} = \inf_{m\in M} \|x_n+m\|,$$

so we can find $m_n \in M$ such that

$$\|x_n+m_n\| \longrightarrow 1.$$

Hence

$$\|\varphi\| \geqslant \frac{\|\varphi(x_n+m_n)\|}{\|x_n+m_n\|}$$

$$= \frac{\|\varphi(x_n)\|}{\|x_n+m_n\|} \quad \text{since } M = \ker(\varphi)$$

$$= \frac{\|\Psi(x_n+M)\|}{\|x_n+m_n\|}$$

$$\longrightarrow \|\Psi\|.$$

This gives the required inequality. ∎

Now we can show that every hyperplane is the kernel of some (not necessarily continuous) linear functional.

<u>Proposition</u>

Let M be a subspace of a normed space X. Then TFAE:

a. M is a hyperplane.

b. $M = \ker(\mu)$ for some nonzero linear functional $\mu : X \to \mathbb{F}$.

<u>Proof:</u>

$b \Rightarrow a$. Suppose that $M = \ker(\mu)$, where μ is a nonzero linear functional on X. Then by the Isomorphism Theorem, $\exists$ linear bijection $\Psi : X/\ker(\mu) \longrightarrow \mathbb{F}$.

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\pi"'] & \mathbb{F} \\ X/\ker(\mu) \arrow[ur, "\Psi"'] & \end{tikzcd}$$

Since $\mathbb{F}$ is 1-dimensional, we conclude that $X/\ker(\mu)$ is also 1-dimensional.

$a \Rightarrow b$. Assume that M is a hyperplane in X. Then X/M is 1-dimensional, so $\exists$ linear bijection $\psi: X/M \to \mathbb{F}$. Set $\varphi = \psi \circ \pi$. Then $\varphi: X \to \mathbb{F}$ is linear.

Exercise: Show $\ker(\varphi) = M$.

Finally, since $\dim(X/M) = 1$, we have $M \subsetneq X$ (why?). Hence $\ker(\varphi) \neq X$, so φ is not the zero functional. $\blacksquare$

The next result is a bit surprising at first glance, but less so when we consider that kernels of linear functionals are hyperplanes, hence are "almost" the entire space except for "one dimension."

<u>Proposition</u>
Let X be a vector space. If $\mu, \nu : X \to \mathbb{F}$ are nonzero linear functionals, then

$$\ker(\mu) = \ker(\nu) \iff \mu = c\nu \text{ for some nonzero } c \in \mathbb{F}.$$

<u>Proof:</u>
$\Rightarrow$ Suppose that $\ker(\mu) = \ker(\nu)$. Since $\mu \neq 0$, $\exists f \in X$

s.t. $\langle f, \mu \rangle \neq 0$, and by rescaling we may assume

$\langle f, \mu \rangle = 1$. Then $f \notin \ker(\mu) = \ker(\nu)$, so $\langle f, \nu \rangle \neq 0$.

Given any $g \in X$, we have

$$\langle g - \langle g, \mu \rangle f, \mu \rangle = \langle g, \mu \rangle - \langle g, \mu \rangle \langle f, \mu \rangle = 0.$$

Therefore $g - \langle g, \mu \rangle f \in \ker(\mu) = \ker(\nu)$. Hence

$$\langle g, \nu \rangle - \langle g, \mu \rangle \langle f, \nu \rangle = \langle g - \langle g, \mu \rangle f, \nu \rangle = 0.$$

Hence

$$\langle g, \nu \rangle = \langle \langle f, \nu \rangle g, \mu \rangle \quad \forall g \in X$$

so

$$\nu = \langle f, \nu \rangle \mu,$$

and, as we said, $\langle f, \nu \rangle \neq 0.$ $\blacksquare$

Now we turn from results valid for general vector spaces to the setting of normed spaces, where we can consider continuity of functionals and closedness of subspaces.

Propositon

If M is a hyperplane in a normed linear space X, then either M is closed or M is dense in X.

Proof:

We are given that X/M is 1-dimensional. Let $\pi: X \longrightarrow X/M$ be the canonical projection. The closure $\overline{M}$ of M is a subspace of X. Since π is linear, $\pi(\overline{M})$ must be a subspace of X/M. But X/M is 1-dimensional, so there are only two possibilities.

First, we could have $\pi(\overline{M}) = \{0+M\}$. In this case, $\overline{M} \subseteq \ker(\pi) = M$, which implies $\overline{M} = M$, so M is closed.

Second, we could have $\pi(\overline{M}) = X/M$. In this case $\overline{M} = X$ (why?), so M is dense in X.

Finally, we obtain a useful characterization of bounded linear functionals on a normed space.

__Proposition__

Let $\mu : X \to \mathbb{F}$ be a linear functional on a normed space X. Then:

$$\mu \text{ is continuous} \iff \ker(\mu) \text{ is closed.}$$

__Proof:__

$\Rightarrow$ Exercise.

$\Leftarrow$ Suppose that $M = \ker(\mu)$ is closed. We know that M is a hyperplane, so $\dim(X/M) = 1$. Let $\pi : X \to X/M$ be the canonical projection. Since π $\ker(\pi) = M$ is closed, a previous result implies that π is continuous.

Now, by the Isomorphism Theorem, $\exists$ linear bijection $\Psi : X/M \to \mathbb{F}$, although the Isomorphism Theorem doesn't tell us whether Ψ is continuous.

$$\begin{tikzcd} X \arrow[r, "\mu"] \arrow[d, "\pi"'] & \mathbb{F} \\ X/M \arrow[ur, "\psi"'] & \end{tikzcd}$$

However, any linear map from a finite-dimensional space to a normed space is continuous (previous exercise), so in fact ψ must be continuous. Therefore $\mu = \psi \circ \pi$ is continuous as well. $\blacksquare$

Alternatively, the Isomorphism Theorem for <u>normed</u> spaces tells us that $\|\psi\| = \|\mu\|$, so μ is bounded.