5. Baire Category

Definition

Let X be a metric space. Then $E \subseteq X$ is <u>nowhere dense</u>, or <u>rare</u>, if $X \setminus \bar{E}$ is dense in X.

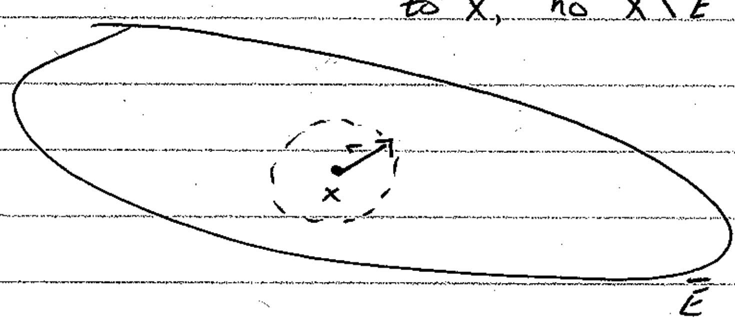
Exercise: Show TFAE:

a. E is nowhere dense

b. $\bar{E}$ contains no interior points

c. $\bar{E}$ contains no open subsets.

$c \Rightarrow a$ is clear: If $\exists$ an open ball $B_r(x)$ in $\bar{E}$, then no element of $X \setminus \bar{E}$ can be closer than r to x, so $X \setminus \bar{E}$ isn't dense.



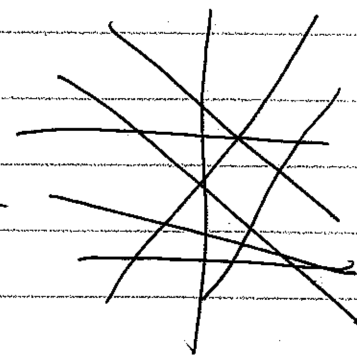
Definition

Let X be a metric space. Then a subset $E \subseteq X$ is <u>meager</u>, or <u>1^{st} category</u>, if it is a countable union of nowhere dense sets.

Examples

$\mathbb{Q}$ in $\mathbb{R}$

union of countably many lines in $\mathbb{R}^2$



Definition

A subset $E \subseteq X$ is <u>nonmeager</u> or <u>2^{nd} category</u> if it is not meager.

A nowhere dense set is very "small" in some sense.

Examples: $\mathbb{Z}$ in $\mathbb{R}$

a line in $\mathbb{R}^2$

Cantor set in $[0,1]$

However, "small" here means something different than small in terms of measure. For example, a variation on the construction of the Cantor set will give a Cantor-like subset C of $[0,1]$ with <u>positive</u> Lebesgue measure, yet C is still nowhere dense — $|C| > 0$ yet $\overline{C} = C$ contains <u>no</u> open subsets.

<u>Example</u>

A countable union of nowhere dense sets is still a "small" set, but need not be nowhere dense.

For example, each singleton $\{x\}$ is nowhere dense in $\mathbb{R}$, but

$$\mathbb{Q} = \bigcup_{r \in \mathbb{Q}} \{r\}$$

is <u>not</u> nowhere dense, as $\overline{\mathbb{Q}} = \mathbb{R}$ is open.

Baire Category Theorem

A nonempty complete metric space is nonmeager (in X)

Consequently, if

$$X = \bigcup_{k=1}^{\infty} E_k$$

with each E_k closed, then at least one E_k contains a nonempty open subset.

<u>Example</u>: $\mathbb{R}^2$ cannot be written as the union of countably many straight lines (this is also easy to prove directly).

<u>Remark</u>

<u>Completeness</u> is critical to the proof, and will be very important in several following results (e.g., the open mapping theorem).

If we do not assume completeness, then the result is false: $\exists$ incomplete metric spaces that are meager in themselves.

Proof:

Let $d(\cdot,\cdot)$ be the distance on X.

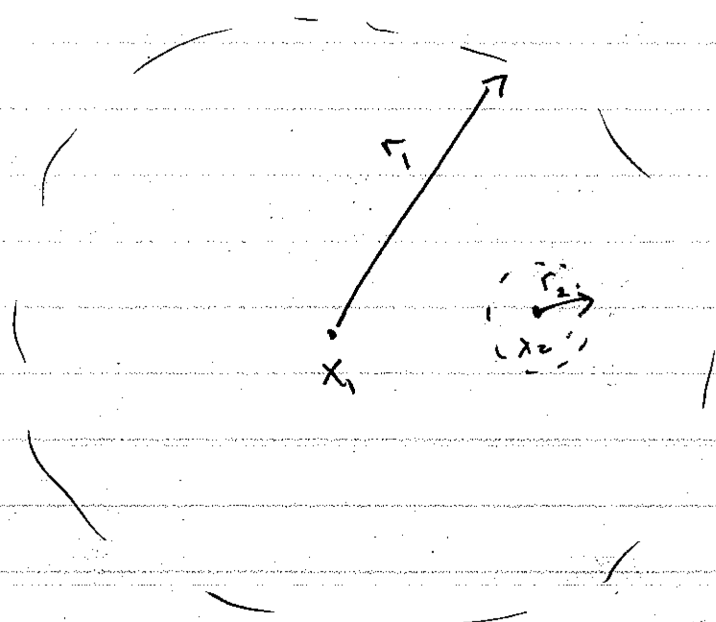
Suppose that $X = \bigcup_{n=1}^{\infty} E_n$ where each E_n is nowhere dense.

Then $U_n = X \setminus \overline{E_n}$ is dense and open.

Choose $x_1 \in U_1$, $B_1 = B_{r_1}(x_1) \subseteq U_1$.

Since U_2 is dense, $\exists\ x_2 \in \underbrace{U_2 \cap B_1}_{\text{open}}$

and hence $\exists\ B_2 = B_{r_2}(x_2) \subseteq U_2 \cap B_1$.



B_1

Take r_2 small enough that

$r_2 < \frac{r_1}{2}$ & $\overline{B_2} \subseteq B_1$

Repeat: get

$$x_n \in U_n, \quad B_n = B_{r_n}(x_n) \subseteq U_n \quad \text{s.t.}$$

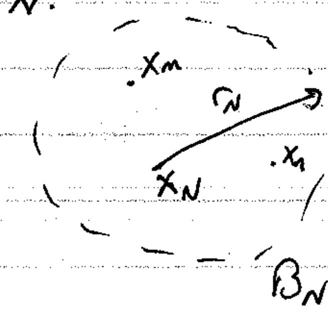
$$\overline{B_n} \subseteq B_{n-1}, \quad r_n < \frac{r_{n-1}}{2}$$

In particular, $r_n \to 0$.

Choose ε. Then $\exists N$ s.t. $r_N < \frac{\varepsilon}{2}$.

If $m, n > N$ then $x_m, x_n \in B_N$.

Hence



$$d(x_m, x_n) < 2r_N < \varepsilon.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is Cauchy.

Since X is complete, $\exists x \in X$ s.t. $x_n \to x$.

Choose any N. Then by nestedness,

$$x_n \in B_{r_{N+1}}(x_{N+1}) \quad \forall n > N$$

& $x_n \to x$, so $x \in \overline{B_{r_{N+1}}(x_{N+1})} \subseteq B_{r_N}(x_N) = B_N$.

This is true for all N, so

Thus
$$x \in \bigcap_{n=1}^{\infty} B_n \subseteq \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} (X \setminus \overline{E_n}).$$

Hence
$$x \notin \bigcup_{n=1}^{\infty} E_n = X,$$

which is a contradiction. ∎

Exercise

Prove that ℓ^1 is a meager subset of ℓ^2 (even though ℓ^1 is dense in ℓ^2).

Hint: Consider $F_N = \left\{ x \in \ell^2 : \sum_{k=1}^{\infty} |x_k| \le N \right\}$.

Exercise

Let $C[0,1]$ be the Banach space of continuous functions on $[0,1]$ under the L^∞-norm. Define

$$F_n = \left\{ f \in C[0,1] : |f(x) - f(y)| \le n|x-y| \quad \forall x, y \in [0,1] \right\}.$$

a. Show that F_n is a closed subset of $C[0,1]$.

b. Show that F_n is nowhere dense in $C[0,1]$.

c. Show that $C'[0,1] = \{ f \in C[0,1] : f' \in C[0,1] \}$

is meager in $C[0,1]$ (even though it is dense in $C[0,1]$).

Exercise

Let D be the subset of $C[0,1]$ consisting of all functions $f \in C[0,1]$ which have a right-hand derivative at at least one point in $[0,1]$.

a. Show that D is meager in $C[0,1]$.

Hint: $F_n = \{ f \in C[0,1] : \exists x_0 \text{ s.t. } |f(x) - f(x_0)| \leq n(x - x_0), \; x_0 \leq x < 1 \}$

b. Show $\exists$ functions in $C[0,1]$ that are not differentiable at any point.