The Uniform Boundedness Principle

Theorem: Uniform Boundedness Principle

Assume:
- $X$ is Banach
- $Y$ is normed
- $A_i \in \mathcal{B}(X,Y)$ for $i \in I$ are given, $I$ arbitrary

Then:

\[ \sup_{i \in I} \|A_i x\| < \infty \quad \forall x \in X \implies \sup_{i \in I} \|A_i\| < \infty. \]

uniformly bounded at each point

uniformly bounded in operator norm

This is a very surprising and strong result!

Compare the difference between the hypothesis and the conclusion!
Hypothesis is that

$$\forall x \in X, \sup_{i \in I} \|A_i x\| < \infty$$

or, by rescaling, this is equivalent to assuming that

$$\forall \|x\| = 1, \sup_{i \in I} \|A_i x\| < \infty.$$ (**) (x)

The conclusion is much stronger: conclusion is

$$\sup_{i \in I} \sup_{\|x\| = 1} \|A_i x\| < \infty.$$ (***)

Without extra hypotheses, there's no way that (x) will imply (**). Somehow the linearity, boundedness of the $A_i$ and the completeness of $X$ allows us to draw the conclusion (***) from the seemingly much weaker assumption (x).
Let's say as again one more time for emphasis.

If we set

\[ M_x = \sup_{i \in I} \| A_i x \| \quad \text{for } x \in X \]

Then the hypothesis of the UBPs is that

\[ M_x < \infty \quad \forall \| x \| = 1. \]

Given arbitrary scalars \( M_x \), there's no way that we could infer from this that

\[ \sup_{\| x \| = 1} M_x < \infty. \quad (\star) \]

Yet the UBPs say not only that \((\star)\) is true, but even more:

\[ M = \sup_{i \in I} \| A_i \| < \infty \quad (\star\star). \]

Equation \((\star\star)\) is even better than \((\star)\) because if we have \((\star\star)\) then

\[ \sup_{\| x \| = 1} M_x = \sup_{\| x \| = 1} \sup_{i \in I} \| A_i x \| \leq \sup_{\| x \| = 1} \sup_{i \in I} \| A_i \| \| x \| = M < \infty. \]
Proof of \( \& \) UBP

Set
\[
E_n = \{ x \in X : \sup_{i \in I} \| A_i x \| \leq n \}.
\]

By hypothesis,
\[
X = \bigcup_{n=1}^{\infty} E_n.
\]

Claim: Each \( E_n \) is closed.

Suppose that \( x_k \in E_n \) \& \( x_k \to x \in X \).
Since \( A_i \) is bounded, we therefore have \( A_i x_k \to A_i x \).

Hence
\[
\| A_i x \| = \lim_{k \to \infty} \| A_i x_k \| \leq n \quad \text{since} \ x_k \in E_n.
\]

Therefore \( x \in E_n \), so \( E_n \) is closed.

By the Baire Category Theorem, since \( X \) is complete, at least one \( E_n \) must contain an open ball
i.e., \( \exists n \in \mathbb{N}, x_0 \in X, r > 0 \) s.t.
\[
B_r(x_0) \subseteq E_n.
\]
Fix any $x \in X$. By rescaling & translating, we can "move" it to be inside $B_{r}(x_0)$.

That is, set

$$Z = x_0 + s \cdot x$$

where $s = \frac{r}{2\|x\|}$.

Since

$$\|Z - x_0\| = \|s \cdot x\| = \frac{s}{2} < r,$$

we do indeed have $Z \in B_{r}(x_0)$. 

\[\square\]
Then $z \in B_r(x_0) \subseteq E_n$, so

$$\sup_{i \in \mathbb{Z}} \| A_i z \| \leq n.$$ 

Hence

$$\| A_i x \| = \| A_i \left( \frac{z - x_0}{s} \right) \|$$

$$\leq \frac{1}{s} \left( \| A_i z \| + \| A_i x_0 \| \right)$$

$$\leq \frac{1}{s} \left( n + n \right)$$

$$= \frac{2 \| x \|}{r} \cdot 2n = \frac{4n}{r} \| x \|.$$ 

This is true \( \forall x \) \& \( \forall i \), so

$$\sup_{i} \| A_i \| \leq \frac{4n}{r} < \infty.$$ 

Corollary: UBP for $Y = xF$

Let $X$ be a Banach space. Then:

$\mathcal{F}$ is a bounded subset of $X^*$ \iff $\forall x \in X$, $\sup_{\mu \in \mathcal{F}} |\langle x, \mu \rangle| < \infty$

Proof:

$\Rightarrow$. Suppose that $\mathcal{F} \subseteq X^*$ is bounded, which means

$$R = \sup_{\mu \in \mathcal{F}} \|\mu\| < \infty.$$  

Then $\forall x \in X$,

$$\sup_{\mu \in \mathcal{F}} |\langle x, \mu \rangle| \leq \sup_{\mu \in \mathcal{F}} \|x\| \|\mu\| \leq R \|x\| < \infty.$$ 

$\Leftarrow$. Assume the RHS holds. Then

$$\mathcal{F} = \{\mu\}_{\mu \in \mathcal{F}} \subseteq X^* = \mathcal{B}(X, xF)$$

satisfies the hypotheses of the UBP. Therefore

$$\sup_{\mu \in \mathcal{F}} \|\mu\| < \infty,$$

which says that $\mathcal{F}$ is a bounded subset of $X^*$. \qed
The preceding corollary applied $\&$ UBP to a family of operators in $X^*$. The next corollary has a similar flavor, but in it we wish to apply $\&$ UBP to a subset of $X$. How can we do this? — Elements of $X$ are not operators. But although $x \in X$ is not an operator, it determines a function $x \in X^{**}$, i.e., an operator on $X^*$. We apply $\&$ UBP to a subset of $X^{**}$ in order to draw a conclusion about $X$.

\textbf{Corollary}

Let $X$ be a normed space and let $E \subseteq X$. Then:

$$E \text{ is a bounded subset of } X \iff \forall \mu \in X^*, \sup_{x \in E} |\langle x, \mu \rangle| < \infty.$$ 

\textbf{Proof}

$\Rightarrow$ Assume that $E \subseteq X$ is bounded. This means that

$$R = \sup_{x \in E} \|x\| < \infty.$$ 

Therefore, if $\mu \in X^*$ then

$$\sup_{x \in E} |\langle x, \mu \rangle| \leq \sup_{x \in E} \|x\| \|\mu\| = R \|\mu\| < \infty.$$
Assume that the RHS holds. Recall that there is a natural embedding of $X$ into $X^{**}$: each $x \in X$ determines a unique element $\hat{x} \in X^{**}$ defined by

$$\langle \mu, \hat{x} \rangle = \langle x, \mu \rangle, \quad \mu \in X^*.$$  

Consider a family of operators

$$\{\hat{x}\}_{x \in E} \subseteq X^{**} = B(X^*, E).$$

Note that $X^*$ is a Banach space. Further, by hypothesis we have for each $\mu \in X^*$ that

$$\sup_{x \in E} |\langle \mu, \hat{x} \rangle| = \sup_{x \in E} |\langle x, \mu \rangle| < \infty.$$

The UBP therefore implies that

$$\sup_{x \in E} \|\hat{x}\| < \infty.$$  

But we know that $\|\hat{x}\| = \|x\|$ (operator norm of $\hat{x}$ equals the norm of the element $x$), so

$$\sup_{x \in E} \|x\| < \infty,$$

and hence $E$ is bounded.
The following result is a special case of the Uniform Boundedness Principle, although some authors use the terms "Uniform Boundedness Principle" and "Banach–Steinhaus Theorem" interchangeably.

**Banach–Steinhaus Theorem**

Let $X, Y$ be Banach spaces. Assume

a. $A_n \in B(X, Y)$ is given for each $n \in \mathbb{N}$, and

b. $x \in X$, ${A_n x}$ converges in $Y$.

Define $A x = \lim_{n \to \infty} A_n x$, $x \in X$. Then

i. $A \in B(X, Y)$,

ii. $\limsup_{n} \|A_n\| < \infty$,

iii. $\|A\| \leq \limsup_{n} \|A_n\|$.

**Remark**

The fact that the sequence ${A_n}$ is a countable sequence of operators is important.

It is surprising that the assumption that $Ax = \lim A_n x$ exists pointwise implies $A$ is bounded!
Proof

Exercise: $A$ is linear (easy)

Since $\{A_n x\}_{n \in \mathbb{N}}$ converges in $Y$, for each individual $x \in X$ we know that

$$\sup_n \|A_n x\| < \infty \quad \forall x \in X.$$ 

The UBP therefore implies that

$$M = \sup_n \|A_n\| < \infty.$$ 

Hence for each $x \in X$ we have

$$\|A x\| \leq \|A x - A_n x\| + \|A_n x\|$$

$$\leq \|A x - A_n x\| + \|A_n\| \|x\|$$

$$\leq \|A x - A_n x\| + M \|x\|$$

$$\rightarrow 0 + M \|x\|, \quad \text{as } n \to \infty.$$ 

Hence $\|A\| \leq M.$

Exercise: Show that we actually have

$$\|A\| \leq \liminf_{n \to \infty} \|A_n\|.$$
As an application, we use Banach–Steinhaus to prove the following result.

**Proposition**

Fix $1 \leq p < \infty$ and let $x = (x_1, x_2, \ldots)$ be any sequence of scalars. Then

$$\sum_{k=1}^{\infty} x_k y_k \text{ converges } \forall y \in \ell^p \iff x \in \ell^{p'}$$

**Proof:**

$\Leftarrow$ Follows immediately from Hölder's Inequality.

$\Rightarrow$ Suppose the left-hand side holds. Define

$$T, \quad T_N : \ell^p \to \ell^1 \text{ by}$$

$$Ty = \sum_{k=1}^{\infty} x_k y_k, \quad T_N y = \sum_{k=1}^{N} x_k y_k.$$

Clearly $T_N$ is linear, and for $y \in \ell^p$ we have

$$|Ty| \leq \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \left( \sum_{k=1}^{\infty} |y_k|^p \right)^{1/p'}$$

$$= \frac{C_N}{N} \left( \sum_{k=1}^{N} |y_k|^p \right)^{1/p}$$

$$\leq C_N \|y\|_p,$$

where $C_N = \left( \sum_{k=1}^{N} |x_k|^p \right)^{1/p}$ is a finite constant independent...
of \( y \) (although not independent of \( N \)). Hence for each \( N \in \mathbb{N} \) we have \( T_N \in B(l^p, l^p) = (l^p)^* \).

Since for each \( y \in l^p \) we have that \( T_N y \rightarrow Ty \)

(as scalars in \( l^p \)), & Banach-Steinhaus

Theorem implies \( \| T \| = \sup N \| T_N \| < \infty \). Since \( (l^p)^* \cong l^p \),

we conclude that \( Z \in l^p \) such that

\[
T y = \sum_{k=1}^{\infty} y_k Z_k , \quad y \in l^p .
\]

But \( \lim x_n = T e_n = Z_n \) for all \( n \), so

\[
X = Z \in l^p .
\]

Exercise: Does the result still hold if \( p = \infty \)?
Solution: $p = a$. Yes, $\mathbb{R}$ is easy directly.

Choose $|y_k| = 1$ s.t. $x_k y_k = |x_k|$. Then

$y \in l^\infty$, so $\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} |x_k|$ converges.

Hence $x \in l^1$. 

Exercise

Fix $1 \leq p, q < \infty$. Suppose that $A = [a_{ij}]_{i,j \in \mathbb{N}}$ is an infinite matrix that satisfies

a. $\forall x \in \ell^p, \forall i \in \mathbb{N}, (Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j$ converges

b. $\forall x \in \ell^p, Ax = ((Ax)_i)_{i \in \mathbb{N}} \in \ell^q$

Prove that $A \in \mathcal{B}(\ell^p, \ell^q)$.

Hint

Let $a_i = (a_{ij})_{j \in \mathbb{N}}$. Apply the preceding exercise to conclude that $a_i \in \ell^p$, for each $i$.

Then consider

$$A_N x = (\sum_{j=1}^{\infty} a_{1j} x_j, \ldots, \sum_{j=1}^{\infty} a_{nj} x_j, 0, 0, 0, \ldots)$$

Exercise: What if $p = \infty$ or $q = \infty$?

Exercise: After we do the Closed Graph Theorem, use it to give another proof of the exercise.
Exercise
Let $X$, $Y$, $Z$ be Banach spaces and let

$B : X \times Y \to Z$ be given. For $x \in X$ and $y \in Y$ define

$B_x : Y \to Z \quad B^y : X \to Z$

$B_x(y) = B(x, y) \quad B^y(x) = B(x, y)$.

We say that $B$ is \underline{bilinear} if $B_x$ and $B^y$ are linear for each $x$ and $y$. Prove that if $B$ is bilinear and continuous, then \exists $M < \infty$ s.t.

$\forall x \in X, \forall y \in Y, \quad \|B(x, y)\| \leq M \|x\| \|y\|$. 
Exercise
A sequence \( \{x_n\}_{n=1}^\infty \) in a Hilbert space \( H \) is a
Bessel sequence if
\[
\forall x \in H, \quad \sum_{n=1}^\infty |\langle x, x_n \rangle|^2 < \infty.
\]

a. Prove that if \( \{x_n\}_{n=1}^\infty \) is a Bessel sequence, then
\[
\exists \text{ finite } B > 0 \text{ s.t. } \forall x \in H, \quad \sum_{n=1}^\infty |\langle x, x_n \rangle|^2 \leq B \|x\|^2.
\]

b. Prove that if \( \{x_n\}_{n=1}^\infty \) is a Bessel, then for any sequence \( \{c_n\}_{n=1}^\infty \in \ell^2 \), the series
\[
\sum_{n=1}^\infty c_n x_n
\]
converges (in the norm of \( H \)), and
\[
\left\| \sum_{n=1}^\infty c_n x_n \right\|^2 \leq B \sum_{n=1}^\infty |c_n|^2.
\]
Hint: Show the series is Cauchy.

c. Show that under the same hypotheses, \( \sum_{n=1}^\infty x_n \) converges unconditionally, i.e., regardless of ordering.
d. After we do the Closed Graph Theorem, give another proof of part a based on Fatou's Lemma instead of the Uniform Boundedness Principle.