The Uniform Boundedness Principle

Theorem: Uniform Boundedness Principle

Assume

X is Banach

Y is normed

$A_i \in \mathcal{B}(X,Y)$ for $i \in I$ are given, I arbitrary

Then:

$$\underbrace{\sup_{i\in I} \|A_i x\| < \infty \quad \forall x \in X}_{\text{uniformly bounded at each point}} \implies \underbrace{\sup_{i \in I} \|A_i\| < \infty.}_{\text{uniformly bounded in \underline{operator} \underline{norm}}}$$

This is a very surprising & strong result! Compare the difference between the hypothesis and the conclusion!

Hypothesis is that

$$\forall x \in X, \quad \sup_{i \in I} \|A_i x\| < \infty$$

or, by rescaling, this is equivalent to assuming that

$$\forall \|x\| = 1, \quad \sup_{i \in I} \|A_i x\| < \infty. \qquad (*)$$

The conclusion is <u>much</u> stronger: conclusion is

$$\sup_{i \in I} \sup_{\|x\|=1} \|A_i x\| < \infty \qquad (**)$$

Without extra hypotheses, there's no way that $(*)$ will imply $(**)$. Somehow the linearity, boundedness of the A_i and the completeness of X allows us to draw the conclusion $(**)$ from the seemingly much weaker assumption $(*)$.

Let's say this again one more time for emphasis. If we set

$$M_x = \sup_{i \in I} \|A_i x\| \quad \text{for } x \in X$$

Then the hypothesis of the UBP is that

$$M_x < \infty \qquad \forall \|x\| = 1.$$

Given arbitrary scalars M_x, there's no way that we could infer from this that

$$\sup_{\|x\|=1} M_x < \infty. \qquad (*)$$

Yet the UBP says not only that $(*)$ is true, but even more:

$$M = \sup_{i \in I} \|A_i\| < \infty \qquad (**).$$

Equation $(**)$ is even better than $(*)$ because if we have $(**)$ then

$$\begin{aligned} \sup_{\|x\|=1} M_x &= \sup_{\|x\|=1} \sup_{i \in I} \|A_i x\| \\ &\le \sup_{\|x\|=1} \sup_{i \in I} \|A_i\| \, \|x\| \\ &= M < \infty. \end{aligned}$$

Proof of the UBP

Set

$$E_n = \{x \in X : \sup_{i \in I} \|A_i x\| \leq n\}.$$

By hypothesis,

$$X = \bigcup_{n=1}^{\infty} E_n.$$

Claim: Each E_n is closed.

Suppose that $x_k \in E_n$ & $x_k \to x \in X$.

Since A_i is bounded, we therefore have $A_i x_k \to A_i x$.

Hence

$$\|A_i x\| = \lim_{k \to \infty} \|A_i x_k\| \leq n \quad \text{since } x_k \in E_n.$$

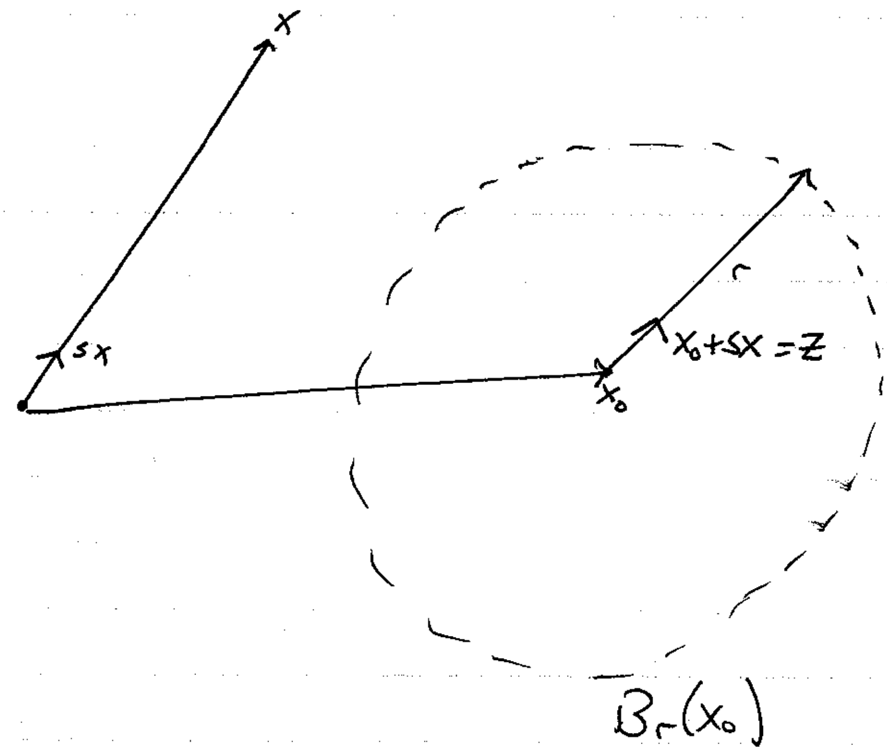
Therefore $x \in E_n$, so E_n is closed.

By the Baire Category Theorem, since X is complete, at least one E_n must contain an open ball

i.e., $\exists\, n \in \mathbb{N}$, $x_0 \in X$, $r > 0$ s.t.

$$B_r(x_0) \subseteq E_n.$$

Fix any $x \in X$. By rescaling & translating, we can "move" it to be inside $B_r(x_0)$



That is, set

$$z = x_0 + sx \quad \text{where} \quad s = \frac{r}{2\|x\|}.$$

Since

$$\|z - x_0\| = \|sx\| = \frac{r}{2} < r,$$

we do indeed have $z \in B_r(x_0)$. ∎

Then $$z \in B_r(x_0) \subseteq E_n,$$

so $$\sup_{i \in I} \|A_i z\| \leq n.$$

Hence

$$\|A_i x\| = \left\| A_i \left(\frac{z - x_0}{s} \right) \right\|$$

$$\leq \frac{1}{s} \left(\|A_i z\| + \|A_i x_0\| \right)$$

$$\leq \frac{1}{s} (n + n)$$

$$= \frac{2\|x\|}{r} \cdot 2n = \frac{4n}{r} \|x\|.$$

This is true $\forall x$ & $\forall i$, so

$$\sup_i \|A_i\| \leq \frac{4n}{r} < \infty. \quad \blacksquare$$

Corollary: UBP for $Y = \mathbb{F}$

Let X be a Banach space. Then:

$$\mathcal{F} \text{ is a bounded subset of } X^* \iff \forall x \in X, \ \sup_{\mu \in \mathcal{F}} |\langle x, \mu \rangle| < \infty$$

Proof:

$\Rightarrow$. Suppose that $\mathcal{F} \subseteq X^*$ is bounded, which means

$$R = \sup_{\mu \in \mathcal{F}} \|\mu\| < \infty.$$

Then $\forall x \in X$,

$$\sup_{\mu \in \mathcal{F}} |\langle x, \mu \rangle| \le \sup_{\mu \in \mathcal{F}} \|x\| \, \|\mu\| \le R\|x\| < \infty.$$

$\Leftarrow$ Assume the RHS holds. Then

$$\mathcal{F} = \{\mu\}_{\mu \in \mathcal{F}} \subseteq X^* = \mathcal{B}(X, \mathbb{F})$$

satisfies the hypotheses of the UBP. Therefore

$$\sup_{\mu \in \mathcal{F}} \|\mu\| < \infty,$$

which says that $\mathcal{F}$ is a bounded subset of X^*. ∎

The preceding corollary applied the UBP to a family of operators in X^*.

The next corollary has a similar flavor, but in it we wish to apply the UBP to a subset of X. How can we do this? - Elements of X are not operators. But although $x \in X$ is not an operator, it determines a function $\hat{x} \in X^{**}$, i.e., an operator on X^*. We apply the UBP to a subset of X^{**} in order to draw a conclusion about X.

<u>Corollary</u>
Let X be a normed space and let $E \subseteq X$. Then:

$$E \text{ is a bounded subset of } X \iff \forall \mu \in X^*, \ \sup_{x \in E} |\langle x, \mu \rangle| < \infty.$$

<u>Proof:</u>
$\Rightarrow$ Assume that $E \subseteq X$ is bounded. This means that

$$R = \sup_{x \in E} \|x\| < \infty.$$

Therefore, if $\mu \in X^*$ then

$$\sup_{x \in E} |\langle x, \mu \rangle| \leq \sup_{x \in E} \|x\| \, \|\mu\| = R\,\|\mu\| < \infty.$$

$\Leftarrow$ Assume that the RHS holds. Recall that there is a natural embedding of X into X^{**}: each $x \in X$ determines a unique element $\hat{x} \in X^{**}$ defined by

$$\langle \mu, \hat{x} \rangle = \langle x, \mu \rangle, \quad \mu \in X^*.$$

Consider the family of operators

$$\{\hat{x}\}_{x \in E} \subseteq X^{**} = \mathcal{B}(X^*, \mathbb{F}).$$

Note that X^* is a Banach space. Further, by hypothesis we have for each $\mu \in X^*$ that

$$\sup_{x \in E} |\langle \mu, \hat{x} \rangle| = \sup_{x \in E} |\langle x, \mu \rangle| < \infty.$$

The UBP therefore implies that

$$\sup_{x \in E} \|\hat{x}\| < \infty.$$

But we know that $\|\hat{x}\| = \|x\|$ (operator norm of $\hat{x}$ equals the norm of the element x), so $\sup_{x \in E} \|x\| < \infty$, and hence E is bounded. $\blacksquare$

The following result is a special case of the Uniform Boundedness Principle, although some authors use the terms "Uniform Boundedness Principle" and "Banach-Steinhaus Theorem" interchangably.

Banach-Steinhaus Theorem

Let X, Y be Banach spaces. Assume

a. $A_n \in \mathcal{B}(X,Y)$ is given for $n \in \mathbb{N}$, and

b. $\forall x \in X$, $\{A_n x\}_{n\in\mathbb{N}}$ converges in Y.

Define $Ax = \lim_{n\to\infty} A_n x$, $x \in X$. Then:

i. $A \in \mathcal{B}(X,Y)$,

ii. $\sup_n \|A_n\| < \infty$,

iii. $\|A\| \leq \sup_n \|A_n\|$.

Remark

The fact that $\{A_n\}_{n\in\mathbb{N}}$ is a countable sequence of operators is important.

It is surprising that the assumption that $Ax = \lim A_n x$ exists pointwise implies A is bounded!

Proof:

Exercise: A is linear (easy).

Since $\{A_n x\}_{n\in\mathbb{N}}$ converges in Y, for each individual $x\in X$ we know that

$$\sup_n \|A_n x\| < \infty \quad \forall x\in X.$$

The UBP therefore implies that

$$M = \sup_n \|A_n\| < \infty.$$

Hence for each $x\in X$ we have

$$\begin{aligned}\|Ax\| &\le \|Ax - A_n x\| + \|A_n x\| \\ &\le \|Ax - A_n x\| + \|A_n\|\,\|x\| \\ &\le \|Ax - A_n x\| + M\|x\| \\ &\longrightarrow 0 + M\|x\| \quad \text{as } n\to\infty\end{aligned}$$

Hence $\|A\| \le M$. ∎

Exercise: Show that we actually have

$$\|A\| \le \liminf_{n\to\infty} \|A_n\|$$

As an application, we use Banach-Steinhaus to prove the following result.

<u>Proposition</u>

Fix $1 \le p < \infty$, and let $x = (x_1, x_2, \dots)$ be any sequence of scalars. Then

$$\sum_{k=1}^{\infty}{}' x_k y_k \text{ converges } \forall y \in \ell^p \iff x \in \ell^{p'}.$$

<u>Proof:</u>

$\Leftarrow$ Follows immediately from Hölder's Inequality.

$\Rightarrow$ Suppose the left-hand side holds. Define

$T, T_N : \ell^p \longrightarrow \mathbb{F}$ by

$$Ty = \sum_{k=1}^{\infty}{}' x_k y_k, \qquad T_N y = \sum_{k=1}^{N}{}' x_k y_k.$$

Clearly T_N is linear, and for $y \in \ell^p$ we have

$$|T_N y| \le \left(\sum_{k=1}^{N}{}' |x_k|^{p'}\right)^{1/p'} \left(\sum_{k=1}^{N}{}' |y_k|^p\right)^{1/p}$$

If $p=1$ then $C_N = \max\{|x_1|, \dots, |x_N|\}$ $\longrightarrow$

$$= C_N \left(\sum_{k=1}^{N}{}' |y_k|^p\right)^{1/p}$$

$$\le C_N \|y\|_p,$$

where $C_N = \left(\sum_{k=1}^{N}{}' |x_k|^{p'}\right)^{1/p'}$ is a finite constant independent of y (although not independent of N). Hence for each $N \in \mathbb{N}$ we have $T_N \in \mathcal{B}(\ell^p, \mathbb{F}) = (\ell^p)^*$.

Since for each $y \in \ell^p$ we have that $T_N y \to Ty$ (as scalars in $\mathbb{F}$), the Banach-Steinhaus Theorem implies that $T \in \mathcal{B}(\ell^p, \mathbb{F}) = (\ell^p)^*$, and $\|T\| \le \sup_N \|T_N\| < \infty$. Since $(\ell^p)^* \cong \ell^{p'}$, we conclude that $\exists\, z \in \ell^{p'}$ such that

$$Ty = \sum_{k=1}^{\infty} y_k z_k, \quad y \in \ell^p.$$

But then $x_n = Te_n = z_n$ for all n, so $x = z \in \ell^{p'}$. ∎

<u>Exercise:</u> Does the result still hold if $p = \infty$?

Solution $p=\infty$. yes, ℓ^∞ is easy directly.

Choose $|y_k| = 1$ s.t. $x_k y_k = |x_k|$. Then

$y \in \ell^\infty$, so $\sum_{k=1}^{\infty} x_k y_k = \sum_{k=1}^{\infty} |x_k|$ converges.

Hence $x \in \ell^1$.

Exercise

Fix $1 \le p, q < \infty$. Suppose that $A = [a_{ij}]_{i,j \in \mathbb{N}}$ is an infinite matrix that satisfies

a. $\forall x \in \ell^p, \forall i \in \mathbb{N}, \quad (Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j$ converges

b. $\forall x \in \ell^p, \quad Ax = ((Ax)_i)_{i \in \mathbb{N}} \in \ell^q$.

Prove that $A \in \mathcal{B}(\ell^p, \ell^q)$.

Hint

Let $a_i = (a_{ij})_{j \in \mathbb{N}}$. Apply the preceding exercise to conclude that $a_i \in \ell^{p'}$ for each i.

Then consider

$$A_N x = \left(\sum_{j=1}^{\infty} a_{1j} x_j, \ldots, \sum_{j=1}^{\infty} a_{Nj} x_j, 0, 0, 0, \ldots \right)$$

Exercise: What if $p = \infty$ or $q = \infty$?

Exercise: After we do the Closed Graph Theorem, use it to give another proof of the exercise.

Exercise

Let X, Y, Z be Banach spaces and let

$B: X \times Y \longrightarrow Z$ be given. For $x \in X$ & $y \in Y$ define

$$B_x : Y \longrightarrow Z \qquad\qquad B^y : X \longrightarrow Z$$

$$B_x(y) = B(x,y) \qquad\qquad B^y(x) = B(x,y).$$

We say that B is <u>bilinear</u> if B_x & B^y are linear for each x & y. Prove that if B is bilinear & continuous, then $\exists M < \infty$ s.t.

$$\forall x \in X, \ \forall y \in Y, \quad \|B(x,y)\| \leq M \|x\| \, \|y\|.$$

Exercise

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is a Bessel sequence if

$$\forall x \in H, \quad \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 < \infty.$$

a. Prove that if $\{x_n\}_{n\in\mathbb{N}}$ is a Bessel sequence, then $\exists$ finite $B>0$ s.t.

$$\forall x \in H, \quad \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

b. Prove that if $\{x_n\}_{n\in\mathbb{N}}$ is Bessel then for any sequence $(c_n)_{n\in\mathbb{N}} \in \ell^2$, the series $\sum_{n=1}^{\infty} c_n x_n$ converges (in the norm of H), and

$$\left\|\sum_{n=1}^{\infty} c_n x_n\right\|^2 \leq B \sum_{n=1}^{\infty} |c_n|^2$$

Hint: Show the series is Cauchy.

c. Show that under the same hypotheses, $\sum_{n=1}^{\infty} x_n$ converges unconditionally, i.e., regardless of ordering.

d. After we do the Closed Graph Theorem, give another proof of part a based on that theorem instead of the Uniform Boundedness Principle.