SCHAUDER BASES:

Introduction and Examples

Basis Implications

ONB (H.S. only) $\Longrightarrow$ Bounded Unconditional Basis $\Longrightarrow$ Unconditional Basis $\Longrightarrow$ Schauder Basis

Converse implications fail, even in a Hilbert space.

In a Hilbert space:

Bounded Unconditional Basis $\Longleftrightarrow$ Riesz Basis $\Longleftrightarrow$ Exact Frame

Spanning and Independence

INDEPENDENCE PROPERTIES

$$
\begin{tikzcd}
 & \text{Minimal} \arrow[r, Rightarrow] & \omega\text{-independent} \arrow[r, Rightarrow] & \text{finitely independent} \\
\text{Schauder Basis} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \\
 & \text{Quasibasis} \arrow[r, Rightarrow] & \text{Property S} \arrow[r, Rightarrow] & \text{Complete} \\
\text{Frame (H.S. only)} \arrow[ur, Rightarrow] & & &
\end{tikzcd}
$$

SPANNING PROPERTIES

DEFINITIONS

3.1 Spanning and Independence in Finite Dimensions

Spanning and independence are clear in finite dimensions. A set $\{f_1, \dots, f_M\}$ of M vectors in an N-dimensional vector space H *spans* H if for each vector $f \in H$ there exist scalars c_i (not necessarily unique) such that $f = c_1 f_1 + \cdots + c_M f_M$. This can only happen if $M \geq N$.

On the other hand, $\{f_1, \dots, f_M\}$ is *linearly independent* if whenever a vector $f \in H$ can be written as $f = c_1 f_1 + \cdots + c_M f_M$, it can only be so written in one way, i.e., the scalars c_i are unique if they exist at all. This can only happen if $M \leq N$.

When both of these happen simultaneously, we have a *basis*. In this case every $f \in H$ can be written as $f = c_1 f_1 + \cdots + c_M f_M$ for a unique choice of scalars c_i. This can only happen if $M = N$.

3.2 Spanning in Infinite Dimensions

For proofs, examples, and more information on bases, convergence of series, and related issues in normed spaces that are discussed in this section, we suggest the references [28], [64], [93], [98], [108], [114].

In a completely arbitrary vector space we can only define *finite* sums of vectors, because to define an infinite series we need a notion of convergence, and this requires a norm or metric or at least a topology. Thus, we define the *finite linear span* of a collection of vectors $\{f_\alpha\}_{\alpha \in J}$ to be

$$\operatorname{span}(\{f_\alpha\}_{\alpha \in J}) = \Big\{ \sum_{i=1}^{N} c_i f_{\alpha_i} : N \in \mathbf{N},\, c_i \in \mathbf{C}, \alpha_i \in J \Big\}.$$

We say that $\{f_\alpha\}_{\alpha \in J}$ *spans* V if the finite span is all of V, i.e., every vector in V equals some finite linear combination of the f_α. We say that $\{f_\alpha\}_{\alpha \in J}$ is a *Hamel basis* if it both spans and is finitely linearly independent, or, equivalently, if every nonzero vector $f \in V$ can be written $f = \sum_{i=1}^{N} c_i f_{\alpha_i}$ for a unique choice of indices $\{\alpha_i\}_{i=1}^{N}$ and nonzero scalars $\{c_i\}_{i=1}^{N}$. For most vector spaces, Hamel bases are only known to exist because of the Axiom of Choice; in fact, the statement "Every vector space has a Hamel basis" is equivalent to the Axiom of Choice. Although Hamel bases are sometimes just called "bases," this is potentially confusing because if V is a normed space, then the word *basis* is usually reserved for something different (see Definition 3).

As soon as we impose a little more structure on our vector space, we can often construct systems which are much more convenient than Hamel bases. For example, in a Banach space we have a norm, and hence can form "infinite linear combinations" by using infinite series. In particular, given a collection $\{f_i\}_{i \in \mathbf{N}}$ indexed by the natural numbers and given scalars $\{c_i\}_{i \in \mathbf{N}}$, we say the series $f = \sum_{i=1}^{\infty} c_i f_i$ converges and equals f if $\|f - \sum_{i=1}^{N} c_i f_i\| \to 0$ as $N \to \infty$. Note that order in this series is important; if we change the order of indices we are not guaranteed that the series will still converge. If the convergence does not depend on the order it is called *unconditional convergence*, otherwise it is *conditional convergence*.

A related but distinct consequence of the fact that we have a norm is that we can form the closure of the finite linear span by constructing the set of all possible limits of finite linear combinations. This set is called the *closed span*, and is denoted $\overline{\mathrm{span}}(\{f_i\}_{i\in\mathbf{N}})$. Given $f \in \overline{\mathrm{span}}(\{f_i\}_{i\in\mathbf{N}})$, by definition there exist vectors $g_N \in \mathrm{span}(\{f_i\}_{i\in\mathbf{N}})$ which converge to f. However, this is not the same as forming infinite linear combinations. While each g_N is some finite linear combination of the f_i, it need not be true that we can write $g_N = \sum_{i=1}^{N} c_i f_i$ using a *single* sequence of scalars $\{c_i\}_{i\in\mathbf{N}}$.

Using these notions, we can form several variations on "spanning sets."

Definition 3. *Let $\{f_i\}_{i\in\mathbf{N}}$ be a countable sequence of vectors in a Banach space X.*

(a) *$\{f_i\}_{i\in\mathbf{N}}$ is* complete *(or* total *or* fundamental*) if $\overline{\mathrm{span}}(\{f_i\}_{i\in\mathbf{N}}) = X$, i.e., for each $f \in X$ and each $N \in \mathbf{N}$ there exist scalars $\{c_{N,i}(f)\}_{i\in\mathbf{N}}$ such that $\sum_{i=1}^{N} c_{N,i}(f)\, f_i \to f$ as $N \to \infty$.*

(b) *$\{f_i\}_{i\in\mathbf{N}}$ has* Property S *if for each $f \in X$ there exist scalars $\{c_i(f)\}_{i\in\mathbf{N}}$ such that*

$$f = \sum_{i=1}^{\infty} c_i(f) f_i. \tag{1}$$

(c) *$\{f_i\}_{i\in\mathbf{N}}$ is a* quasibasis *if it has Property S and for each $i \in \mathbf{N}$ the mapping $f \mapsto c_i(f)$ is linear and continuous (and hence defines an element of the dual space X^*).*

(d) *$\{f_i\}_{i\in\mathbf{N}}$ is a* basis *or* Schauder basis *if it has Property S and for each $f \in X$ the scalars $\{c_i(f)\}_{i\in\mathbf{N}}$ are unique.*

Completeness is a weak property. The definition says that there are finite linear combinations of the f_i that converge to f, but the scalars needed can change completely as the length N of the linear combination increases. On the other hand, unlike the other properties there exists a nice, simple characterization of complete sequences. For the case of a Hilbert space it is:

$$\{f_i\}_{i\in\mathbf{N}} \text{ is complete} \quad \Longleftrightarrow \quad \text{only } f = 0 \text{ is orthogonal to every } f_i$$

(for a general Banach space we just have to take f to lie in the dual space X^*). Consequently, if $\{f_i\}_{i\in\mathbf{N}}$ is complete, then every $f \in H$ is uniquely determined by the sequence of inner products $\{\langle f, f_i\rangle\}_{i\in\mathbf{N}}$, or in other words, the *analysis operator* $T(f) = \{\langle f, f_i\rangle\}_{i\in\mathbf{N}}$ is an injective mapping into the space of all sequences. However, this doesn't give us an algorithm for constructing f from those inner products, and in general there need not exist a stable way to do so, i.e., T^{-1} need not be continuous.

Property S seems to have no standard name in the literature (hence the uncreative name invented here), perhaps because it is not really a very useful concept by itself. In particular, the definition fails to provide us with a stable algorithm for finding a choice of coefficients $c_i(f)$ that can be used to represent f. The definition of quasibasis addresses this somewhat by requiring that each mapping $f \mapsto c_i(f)$ be continuous (for more details on quasibases, see [86] and the references therein). However, this is still not sufficient in most applications, as it is not so much the continuity of each individual map $f \mapsto c_i(f)$ that is important but rather the continuity of the mapping from f to the entire associated sequence $\{c_i(f)\}_{i \in \mathbf{N}}$. In other words, in concrete applications there is often some particular associated Banach space X_d of sequences (imposed by the context), and the mapping $f \mapsto \{c_i(f)\}_{i \in \mathbf{N}}$ must be a continuous linear map of X into X_d. Specializing to the Hilbert space case, this is one of the ideas behind the definition of *frames* (see Section 3.3).

Imposing uniqueness seems to be a natural requirement, and in fact, it can be shown that even though the definition of basis does not include the requirement that $f \mapsto c_i(f)$ be continuous, this follows automatically from the uniqueness assumption (and the fact that we are using *norm convergence*). Thus every basis is actually a quasibasis. Unfortunately, in many contexts uniqueness is simply too restrictive. For example, this is the case for Gabor systems (compare the Balian–Low Theorem, Theorem 16 below). The terms "basis" and "Schauder basis" are used interchangeably in the Banach space setting.

We can summarize the relations among the "spanning type" properties introduced so far by the following implications:

$$\text{basis} \;\; \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} \;\; \text{quasibasis} \;\; \Longrightarrow \;\; \text{Property S} \;\; \begin{array}{c}\Longrightarrow \\ \not\Longleftarrow\end{array} \;\; \text{complete.}$$

It seems unclear whether every system with Property S must actually be a quasibasis (compare [54]), but the other implications are known to not be reversible in general (even in a Hilbert space).

3.4 Independence in Infinite Dimensions

We explore independence in more detail in this section. For proofs and more information, see [28], [64], [93], [98], [108], [114].

The following are several shades of gray in the possible definition of independence.

Definition 7. *Let $\{f_i\}_{i\in\mathbf{N}}$ be a countable sequence of elements in a Banach space X.*

(a) $\{f_i\}_{i\in\mathbf{N}}$ *is a* basis *or* Schauder basis *if for each $f \in X$ there exist unique scalars c_i such that $f = \sum_i c_i f_i$.*

(b) $\{f_i\}_{i\in\mathbf{N}}$ *is* minimal *if for each $j \in \mathbf{N}$, the vector f_j does not lie in $\overline{\operatorname{span}}(\{f_i\}_{i\neq j})$. Equivalently (via Hahn–Banach), there must exist a sequence $\{\tilde{f}_i\}_{i\in\mathbf{N}}$ in the dual space X^* that is* biorthogonal *to $\{f_i\}_{i\in\mathbf{N}}$, i.e., $\langle f_i, \tilde{f}_j\rangle = 1$ if $i = j$ and 0 if $i \neq j$.*

(c) $\{f_i\}_{i\in\mathbf{N}}$ *is* ω-independent *if the series $\sum_{i=1}^{\infty} c_i f_i$ can converge and equal the zero vector only when every $c_i = 0$.*

(d) $\{f_i\}_{i\in\mathbf{N}}$ *is* finitely independent *(or simply* independent*) if every finite subset is independent, i.e., for any N we have $\sum_{i=1}^{N} c_i f_i = 0$ if and only if $c_1 = \cdots = c_N = 0$.*

For example, consider the system of exponentials $\{e_{n\beta}\}_{n\in\mathbf{Z}}$ described in Example 5. We have already seen that the system is a basis only for $\beta = 1$. If $\beta > 1$ then it is not even complete, while if $\beta < 1$ then it is not a basis because we showed explicitly in (5) that the vector e_0 has two different series representations. Additionally, equation (6) implies that e_0 lies in the closure of $\operatorname{span}(\{e_{n\beta}\}_{n\neq 0})$, so the system is not minimal. Further, by subtracting e_0 from both sides of (6) we obtain a nontrivial infinite series that converges and equals the zero vector, so the system is not ω-independent. Even so, that system is finitely independent.

The following implications among these properties hold, none of which is reversible in general (even in a Hilbert space):

$$\text{basis} \;\; \substack{\Longrightarrow \\ \not\Longleftarrow} \;\; \text{minimal} \;\; \substack{\Longrightarrow \\ \not\Longleftarrow} \;\; \omega\text{-independent} \;\; \substack{\Longrightarrow \\ \not\Longleftarrow} \;\; \text{finitely independent}.$$

One technical point is that the definition of basis really combines aspects of both spanning and independence, i.e., a basis is necessarily complete and has Property S. Adding completeness doesn't change the implications above, e.g., every basis is both minimal and complete, but a minimal sequence that is complete need not be a basis (a sequence which is both minimal and complete is sometimes called an *exact sequence*). On the other hand, Property S is exactly what is missing for a minimal or ω-independent sequence to be a basis, for with either of those hypotheses, once we know that an infinite series $\sum_{i=1}^{\infty} c_i f_i$ converges, we can conclude that the coefficients are unique. However, as shown by the example of the exponentials, finite independence combined with Property S is *not* sufficient to ensure that we have a basis. Thus we have the following equivalences:

$$\text{basis} \iff \begin{array}{c}\text{minimal with}\\ \text{Property S}\end{array} \iff \begin{array}{c}\omega\text{-independent}\\ \text{with Property S,}\end{array}$$

and each of these implies finite independence, but not conversely: a finitely independent sequence which has Property S need not be a basis. Similarly, combining the various independence criteria with a frame hypothesis, we obtain the following result, which should be compared to Example 5, where we showed that a frame which is finitely independent need not be a basis.

DEFINITIONS AGAIN

4. BASES IN BANACH SPACES

Since a Banach space X is a vector space, it must possess a *Hamel*, or *vector space*, *basis*, i.e., a subset $\{x_\gamma\}_{\gamma\in\Gamma}$ whose finite linear span is all of X and which has the property that every finite subcollection is linearly independent. Any element $x \in X$ can therefore be written as some *finite* linear combination of x_γ. However, even a separable infinite-dimensional Banach space would require an *uncountable* Hamel basis. Moreover, the proof of the existence of Hamel bases for arbitrary infinite-dimensional spaces requires the Axiom of Choice (in fact, in can be shown that the statement "Every vector space has a Hamel basis" is equivalent to the Axiom of Choice). Hence for most Banach spaces there is no constructive method of producing a Hamel basis.

Example 4.1. [Gol66, p. 101]. We will use the existence of Hamel bases to show that if X is an infinite-dimensional Banach space, then there exist linear functionals on X which are not continuous. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a Hamel basis for an infinite-dimensional Banach space X, normalized so that $\|x_\gamma\| = 1$ for every $\gamma \in \Gamma$. Let $\Gamma_0 = \{\gamma_1, \gamma_2, \dots\}$ be any countable subsequence of Γ. Define $\mu: X \to \mathbf{C}$ by setting $\mu(\gamma_n) = n$ for $n \in \mathbf{N}$ and $\mu(\gamma) = 0$ for $\gamma \in \Gamma\backslash\Gamma_0$, and then extending μ linearly to X. Then this μ is a linear functional on X, but it is not bounded. $\diamondsuit$

More useful than a Hamel basis is a *countable* sequence $\{x_n\}$ such that every element $x \in X$ can be written as some unique *infinite* linear combination $x = \sum c_n x_n$. This leads to the following definition.

Definition 4.2.

(a) A sequence $\{x_n\}$ in a Banach space X is a *basis* for X if

$$\forall\, x \in X, \quad \exists \text{ unique scalars } a_n(x) \text{ such that } x \;=\; \sum_n a_n(x)\, x_n. \tag{4.1}$$

(b) A basis $\{x_n\}$ is an *unconditional basis* if the series in (4.1) converges unconditionally for each $x \in X$.

(c) A basis $\{x_n\}$ is an *absolutely convergent basis* if the series in (4.1) converges absolutely for each $x \in X$.

(d) A basis $\{x_n\}$ is a *bounded basis* if $\{x_n\}$ is norm-bounded both above and below, i.e., if $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$.

(e) A basis $\{x_n\}$ is a *normalized basis* if $\{x_n\}$ is normalized, i.e, if $\|x_n\| = 1$ for every n. $\diamondsuit$

Absolutely convergent bases are studied in detail in Chapter 5. Unconditional bases are studied in detail in Chapter 9.

Note that if $\{x_n\}$ is a basis, then the fact that each $x \in X$ can be written uniquely as $x = \sum a_n(x)\, x_n$ implies that $x_n \neq 0$ for every n. As a consequence, $\{x_n/\|x_n\|\}$ is a normalized basis for X.

Exercise: If X has an absolutely convergent basis, then $\exists$ topological isomorphism $T: X \to \ell^1$ (and conversely).

If X possesses a basis $\{x_n\}$ then X must be separable, since the set of all finite linear combinations $\sum_{n=1}^{N} c_n x_n$ with rational c_n (or rational real and imaginary parts if the c_n are complex) forms a countable, dense subset of X. The question of whether every separable Banach space possesses a basis was a longstanding problem known as the *Basis Problem.* It was shown by Enflo [Enf73] that there do exist separable, reflexive Banach spaces which do not possess any bases.

Notation 4.3. Note that the coefficients $a_n(x)$ defined in (4.1) are linear functions of x. Moreover, they are uniquely determined by the basis, i.e., the basis $\{x_n\}$ determines a unique collection of linear functionals $a_n: X \to \mathbf{F}$. We therefore call $\{a_n\}$ the *associated sequence of coefficient functionals.* Since these functionals are uniquely determined, we often do not declare them explicitly. When we do need to refer explicitly to both the basis and the associated coefficient functionals, we will write "$(\{x_n\}, \{a_n\})$ is a basis" to mean that $\{x_n\}$ is a basis with associated coefficient functionals $\{a_n\}$. We show in Theorem 4.11 that the coefficient functionals for any basis must be continuous, i.e., $\{a_n\} \subset X^*$.

Further, note that since $x_m = \sum a_n(x)\, x_n$ and $x_m = \sum \delta_{mn}\, x_n$ are two expansions of x_m, we must have $a_m(x_n) = \delta_{mn}$ for every m and n. We therefore say that the sequences $\{x_n\} \subset X$ and $\{a_n\} \subset X^*$ are *biorthogonal,* and we often say that $\{a_n\}$ is the *biorthogonal system* associated with $\{x_n\}$. General biorthogonal systems are considered in more detail in Chapter 7. In particular, we show there that the fact that $\{x_n\}$ is a basis implies that $\{a_n\}$ is the unique sequence in X^* that is biorthogonal to $\{x_n\}$. $\diamondsuit$

Example 4.4. Fix $1 \le p < \infty$, and consider the space $X = \ell^p$ defined in Example 1.6. Define sequences $e_n = (\delta_{mn})_{m=1}^{\infty} = (0, \dots, 0, 1, 0, \dots)$, where the 1 is in the nth position. Then $\{e_n\}$ is a basis for ℓ^p, often called the *standard basis* for ℓ^p. Note that $\{e_n\}$ is its own sequence of coefficient functionals.

On the other hand, $\{e_n\}$ is not a basis for ℓ^∞, and indeed ℓ^∞ has no bases whatsoever since it is not separable. Using the ℓ^∞ norm, the sequence $\{e_n\}$ is a basis for the space c_0 defined in Example 1.6(c). $\diamondsuit$

We are primarily interested in bases for which the coefficient functionals $\{a_n\}$ are *continuous.* We therefore give such bases a special name.

Definition 4.5. A basis $(\{x_n\}, \{a_n\})$ is a *Schauder basis* if each coefficient functional a_n is continuous. In this case, each a_n is an element of the dual space, i.e., $a_n \in X^*$ for every n. $\diamondsuit$

We shall see in Theorem 4.11 that *every* basis is a Schauder basis, i.e., the coefficient functionals a_n are *always* continuous. First, however, we require some definitions and miscellaneous facts. In particular, the following operators play a key role in analyzing bases.

Notation 4.6. The *partial sum operators,* or the *natural projections,* associated with the basis $(\{x_n\}, \{a_n\})$ are the mappings $S_N: X \to X$ defined by

$$S_N x = \sum_{n=1}^{N} a_n(x)\, x_n. \quad \diamondsuit$$

DEFINITIONS AND EXAMPLES

6. SOME TYPES OF LINEAR INDEPENDENCE OF SEQUENCES

In an infinite-dimensional Banach space, there are several possible types of linear independence of sequences. We list three of these in the following definition. We will consider minimal sequences in particular in more detail in Chapter 7.

Definition 6.1. A sequence $\{x_n\}$ in a Banach space X is:

(a) *finitely independent* if $\sum_{n=1}^{N} c_n x_n = 0$ implies $c_1 = \cdots = c_N = 0$,

(b) *ω-independent* if $\sum_{n=1}^{\infty} c_n x_n$ converges and equals 0 only when $c_n = 0$ for every n,

(c) *minimal* if $x_m \notin \overline{\text{span}}\{x_n\}_{n\neq m}$ for every m. $\diamondsuit$

Theorem 6.2. *Let $\{x_n\}$ be a sequence in a Banach space X. Then:*

(a) *$\{x_n\}$ is a basis $\implies$ $\{x_n\}$ is minimal and complete.*

(b) *$\{x_n\}$ is minimal $\implies$ $\{x_n\}$ is ω-independent.*

(c) *$\{x_n\}$ is ω-independent $\implies$ $\{x_n\}$ is finitely independent.*

Proof. (a) Assume that $(\{x_n\}, \{a_n\})$ is a basis for a Banach space X. Then $\{x_n\}$ is certainly complete, so we need only show that it is minimal. Fix m, and define $E = \text{span}\{x_n\}_{n\neq m}$. Then, since $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have $\langle x, a_m\rangle = 0$ for every $x \in E$. Since a_m is continuous, this implies $\langle x, a_m\rangle = 0$ for every $x \in \bar{E} = \overline{\text{span}}\{x_n\}_{n\neq m}$. However, we know that $\langle x_m, a_m\rangle = 1$, so we conclude that $x_m \notin \bar{E}$. Hence $\{x_n\}$ is minimal.

(b) Suppose that $\{x_n\}$ is minimal and that $\sum c_n x_n$ converges and equals 0. Let m be such that $c_m \neq 0$. Then $x_m = -\frac{1}{c_m}\sum_{m\neq n} c_n x_n \in \overline{\text{span}}\{x_n\}_{n\neq m}$, a contradiction.

(c) Clear. $\square$

None of the implications in Theorem 6.2 are reversible, as the following examples show.

Example 6.3. [Sin70, p. 24]. *Minimal and complete $\not\Longrightarrow$ basis.*

Define $C(\mathbf{T}) = \{f \in C(\mathbf{R}) : f(t+1) = f(t)\}$, the space of all continuous, 1-periodic functions. Then $C(\mathbf{T})$ is a Banach space under the uniform norm $\|\cdot\|_{L^\infty}$. Consider the functions $e_n(t) = e^{2\pi i n t}$ for $n \in \mathbf{Z}$. Not only are these functions elements of $C(\mathbf{T})$, but they define continuous linear functionals on $C(\mathbf{T})$ via the inner product $\langle f, e_n\rangle = \int_0^1 f(t)\, e^{-2\pi i n t}\, dt$. Further, $\{e_n\}_{n\in\mathbf{Z}}$ is its own biorthogonal system since $\langle e_m, e_n\rangle = \delta_{mn}$. Lemma 7.2 below therefore implies that $\{e_n\}_{n\in\mathbf{Z}}$ is minimal in $C(\mathbf{T})$. The Weierstrass Approximation Theorem [Kat68, p. 15] states that if $f \in C(\mathbf{T})$ then $\left\|f - \sum_{n=-N}^{N} c_n e_n\right\|_{L^\infty} < \varepsilon$ for some scalars c_n. Hence $\text{span}\{e_n\}_{n\in\mathbf{Z}}$ is dense in $C(\mathbf{T})$, and therefore $\{e_n\}_{n\in\mathbf{Z}}$ is complete in $C(\mathbf{T})$. Alternatively, we can demonstrate the completeness as follows. Suppose that $f \in C(\mathbf{T})$ satisfies $\langle f, e_n\rangle = 0$ for every n. Since $C(\mathbf{T}) \subset L^2(\mathbf{T})$ and since $\{e_n\}_{n\in\mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, this implies that f is the zero function in the space

$L^2(\mathbf{T})$, hence is zero almost everywhere. Since f is continuous, it follows that $f(t) = 0$ for all t. Hence $\{e_n\}_{n\in\mathbf{Z}}$ is complete both $C(\mathbf{T})$ and $L^2(\mathbf{T})$ by Corollary 1.41.

Thus, $\{e_n\}_{n\in\mathbf{Z}}$ is both minimal and complete in $C(\mathbf{T})$. Further, if $f = \sum c_n e_n$ converges in $C(\mathbf{T})$, then it is easy to see from the orthonormality of the e_n that $c_n = \langle f, e_n\rangle$. However, it is known that there exist continuous functions $f \in C(\mathbf{T})$ whose Fourier series $f = \sum \langle f, e_n\rangle\, e_n$ do not converge uniformly [Kat68, p. 51]. Therefore, $\{e_n\}_{n\in\mathbf{Z}}$ cannot be a basis for $C(\mathbf{T})$. $\diamondsuit$

Example 6.4. [Sin70, p. 24]. *ω-independent $\not\Rightarrow$ minimal.*

Let X be a Banach space such that there exists a sequence $\{x_n\}$ that is both minimal and complete in X but is not a basis for X (for example, we could use $X = C(\mathbf{T})$ and $x_n(t) = e_n(t) = e^{2\pi int}$ as in Example 6.3). Since $\{x_n\}$ is minimal, it follows from Lemma 7.2 that there exists a sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$. Since $\{x_n\}$ is not a basis, there must exist some $y \in X$ such that the series $\sum \langle y, a_n\rangle\, x_n$ does not converge in X. Consider the sequence $\{y\} \cup \{x_n\}$. This new sequence is certainly complete, and since $y \in \overline{\mathrm{span}}\{x_n\}$, it cannot be minimal. However, we will show that $\{y\} \cup \{x_n\}$ is ω-independent. Assume that $cy + \sum c_n x_n = 0$, i.e., the summation converges and equals zero. If $c \neq 0$ then we would have $y = -\frac{1}{c}\sum c_n x_n$. The biorthogonality of $\{x_n\}$ and $\{a_n\}$ then implies that $\langle y, a_n\rangle = -c_n/c$. But then $\sum \langle y, a_n\rangle\, x_n$ converges, which is a contradiction. Therefore, we must have $c = 0$, and therefore $\sum c_n x_n = 0$. However, $\{x_n\}$ is minimal, and therefore is ω-independent, so this implies that every c_n is zero. Thus $\{y\} \cup \{x_n\}$ is ω-independent and complete, but not minimal.

Alternatively, we can give a Hilbert space example of a complete ω-independent sequence that is not minimal [VD97]. Let $\{e_n\}$ be any orthonormal basis for any separable Hilbert space H, and define $f_1 = e_1$ and $f_n = e_1 + e_n/n$ for $n \geq 2$. Then $\{f_n\}$ is certainly complete since $\mathrm{span}\{f_n\} = \mathrm{span}\{e_n\}$. However, $\|f_1 - f_n\| = \|e_n/n\| = 1/n \to 0$. Therefore $f_1 \in \overline{\mathrm{span}}\{f_n\}_{n\geq 2}$, so $\{f_n\}$ is not minimal. To see that $\{f_n\}$ is ω-independent, suppose that $\sum c_n f_n$ converges and equals zero. Then

$$\sum_{n=1}^{N} c_n f_n \;=\; \left(\sum_{n=1}^{N} c_n\right) e_1 \;+\; \sum_{n=2}^{N} c_n e_n \;\to\; 0.$$

Therefore,

$$\left\|\left(\sum_{n=1}^{N} c_n\right) e_1 \;+\; \sum_{n=2}^{N} c_n e_n\right\|^2 \;=\; \left|\sum_{n=1}^{N} c_n\right|^2 \;+\; \sum_{n=2}^{N} |c_n|^2 \;\to\; 0.$$

This implies immediately that $c_n = 0$ for each $n \geq 2$, and therefore $c_1 = 0$ as well. $\diamondsuit$

Example 6.5. [Sin70, p. 25]. *Finitely independent $\not\Rightarrow$ ω-independent.*

Let $(\{x_n\}, \{a_n\})$ be a basis for a Banach space X, and let $x \in X$ be any element such that $\langle x, a_n\rangle \neq 0$ for every n. For example, we could take $x = \sum \frac{x_n}{2^n\,\|x_n\|}$. Note that x cannot equal any x_n because $\langle x_n, a_m\rangle = 0$ when $m \neq n$. Consider then the new sequence $\{x\} \cup \{x_n\}$. This is certainly complete, and $-x + \sum \langle x, a_n\rangle\, x_n = 0$, so it is not ω-independent. However, we will show that it is finitely independent. Suppose that $cx + \sum_{n=1}^{N} c_n x_n = 0$. Substituting the fact that

$x = \sum \langle x, a_n \rangle\, x_n$, it follows that

$$\sum_{n=1}^{N} \bigl(c\langle x, a_n\rangle + c_n\bigr)\, x_n \;+\; \sum_{n=N+1}^{\infty} c\langle x, a_n\rangle\, x_n \;=\; 0.$$

However, $\{x_n\}$ is a basis, so this is only possible if $c\langle x, a_n\rangle + c_n = 0$ for $n = 1, \dots, N$ and $c\langle x, a_n\rangle = 0$ for $n > N$. Since no $\langle x, a_n\rangle$ is zero we therefore must have $c = 0$. But then $c_1 = \cdots = c_N = 0$, so $\{x\} \cup \{x_n\}$ is finitely independent. $\diamondsuit$

The coefficient functionals for a Schauder basis are continuous!

SCHAUDER BASES

Definition 1. Let X be a Banach space. A sequence $\{e_k\}_{k\in\mathbb{N}}$ in X is a *basis* for X if for each $x \in X$ there exist unique scalars $a_k(x)$ such that

$$x = \sum_{k=1}^{\infty} a_k(x)e_k,$$

where the series converges in the norm of X.

The functionals a_k are linear. If they are also continuous, then $\{e_k\}_{k\in\mathbb{N}}$ is called a *Schauder basis* for X.

The *partial sum operators* associated with the basis are the mappings $S_N\colon X \to X$ defined by

$$S_N x = \sum_{k=1}^{N} a_k(x)e_k.$$

Note that, by definition, $S_N x \to x$ as $N \to \infty$.

This exercise will show that if $\{e_n\}_{n\in\mathbb{N}}$ is a basis for a Banach space X, then it is a Schauder basis. Thus, the mere existence and uniqueness of the functionals a_k implies that they are bounded. There are several essentially similar ways to prove this; feel free to take a different approach.

NOTE: Keep in mind throughout that we do NOT know that the functionals a_k or the operators S_N are bounded; this is exactly what we are trying to prove.

a. Prove that $\{e_k\}_{k\in\mathbb{N}}$ and $\{a_k\}_{k\in\mathbb{N}}$ are biorthogonal, i.e.,

$$a_j(e_k) = \delta_{jk},$$

and that

$$\mathrm{range}(S_N) = E_N = \mathrm{span}\{e_1, \dots, e_N\}.$$

b. Define

$$|||x||| = \sup_N \|S_N x\|, \qquad x \in X.$$

Show that $|||\cdot|||$ is a norm, and that $\|\cdot\| \le |||\cdot|||$.

c. Our next goal is to show that X is complete with respect to $|||\cdot|||$. We break this into several steps.

Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X with respect to $|||\cdot|||$. Show that for each $N \in \mathbb{N}$, there exists $z_N \in E_N$ such that $\|z_N - S_N x_n\| \to 0$ as $n \to \infty$.

d. Show that

$$\lim_{n\to\infty} \Bigl(\sup_N \|z_N - S_N x_n\|\Bigr) = 0.$$

e. Show that $\{z_N\}_{N\in\mathbb{N}}$ is Cauchy with respect to $\|\cdot\|$.
Hint: 3ε argument.

f. Show that there exist scalars $(c_k)_{k\in\mathbb{N}}$, with c_k independent of N, such that $z_N = \sum_{k=1}^{N} c_k e_k$.

Hint: Show that $S_N z_{N+1} = z_N$.
Hint: $S_N|_{E_{N+1}}$ is continuous because S_N is linear and E_{N+1} is finite-dimensional.

g. Show that $x = \sum_{k=1}^{\infty} c_k e_k$ converges with respect to $\|\cdot\|$.

h. Show that $|||x - x_n||| \to 0$, and conclude that X is complete with respect to $|||\cdot|||$.

i. Show that $\|\cdot\|$ and $|||\cdot|||$ are equivalent norms on X, and that each S_N is bounded on X with respect to these norms.

j. Show that each a_k is bounded.
Hint: Consider $S_k - S_{k-1}$.

Hilbert space example that
minimal + complete $\not\Rightarrow$ Schauder basis

Exercise

Let
$$f_n(x) = x e^{2\pi i n x}, \quad n \in \mathbb{Z}$$

a. Show that $\{x e^{2\pi i n x}\}_{n \neq 0}$ is both minimal and complete in $L^2[-\frac{1}{2}, \frac{1}{2}]$, and find the biorthogonal system $\{g_n\}_{n \neq 0}$.

b. Show that $\{x e^{2\pi i n x}\}_{n \neq 0}$ is not a Schauder basis for $L^2[-\frac{1}{2}, \frac{1}{2}]$.

Hint: If it was, then we would have
$$1 \leq \|f_n\|_2 \, \|g_n\|_2 \leq 2C$$
for some finite C (the "basis constant") — we'll prove this fact later

Some facts about unconditional bases in Hilbert spaces (more when we discuss frames)

Exercise

$\{e_n\}_{n\in\mathbb{N}}$ ONB for H.

$T: H \to H$ topological isomorphism.

a. Show $\{Te_n\}_{n\in\mathbb{N}}$ is an unconditional basis for H.

b. What is the biorthogonal system?

c. Give an example showing that $\{Te_n\}_{n\in\mathbb{N}}$ need not be ON.

d. Show that $\{Te_n\}_{n\in\mathbb{N}}$ is bounded in the sense that

$$0 < \inf_n \|Te_n\| \le \sup_n \|Te_n\| < \infty.$$

e. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is any bounded unconditional basis for H. Show $\exists$ ONB $\{e_n\}_{n\in\mathbb{N}}$ & a topological isomorphism $T: H \to H$ s.t. $Te_n = f_n$ for all n.

Remark: This is easier to do after we discuss frames.

Hilbert space example that

Schauder basis $\not\Rightarrow$ Unconditional Basis

<u>Exercise</u>

Let
$$f_n(x) = |x|^{1/4} e^{2\pi i n x}, \quad n \in \mathbb{Z}.$$

a. Show that $\{f_n\}_{n\in\mathbb{Z}}$ is both minimal & complete in $L^2[-\frac{1}{2},\frac{1}{2}]$, and find the biorthogonal sequence $\{g_n\}_{n\in\mathbb{Z}}$

Fact (not easy): $\{f_n\}_{n\in\mathbb{Z}}$ is a Schauder basis for $L^2[-\frac{1}{2},\frac{1}{2}]$.

b. Show that $\{f_n\}_{n\in\mathbb{Z}}$ cannot be an unconditional basis for $L^2[-\frac{1}{2},\frac{1}{2}]$ More generally, $\{g(x)\, e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2[-\frac{1}{2},\frac{1}{2}]$ if & only if

$\exists A, B > 0$ s.t. $\quad A \le |g(x)| \le B$ a.e.