4. BASES IN BANACH SPACES

Since a Banach space X is a vector space, it must possess a *Hamel*, or *vector space*, *basis*, i.e., a subset $\{x_\gamma\}_{\gamma\in\Gamma}$ whose finite linear span is all of X and which has the property that every finite subcollection is linearly independent. Any element $x \in X$ can therefore be written as some *finite* linear combination of x_γ. However, even a separable infinite-dimensional Banach space would require an *uncountable* Hamel basis. Moreover, the proof of the existence of Hamel bases for arbitrary infinite-dimensional spaces requires the Axiom of Choice (in fact, in can be shown that the statement "Every vector space has a Hamel basis" is equivalent to the Axiom of Choice). Hence for most Banach spaces there is no constructive method of producing a Hamel basis.

Example 4.1. [Gol66, p. 101]. We will use the existence of Hamel bases to show that if X is an infinite-dimensional Banach space, then there exist linear functionals on X which are not continuous. Let $\{x_\gamma\}_{\gamma\in\Gamma}$ be a Hamel basis for an infinite-dimensional Banach space X, normalized so that $\|x_\gamma\| = 1$ for every $\gamma \in \Gamma$. Let $\Gamma_0 = \{\gamma_1, \gamma_2, \dots\}$ be any countable subsequence of Γ. Define $\mu: X \to \mathbf{C}$ by setting $\mu(\gamma_n) = n$ for $n \in \mathbf{N}$ and $\mu(\gamma) = 0$ for $\gamma \in \Gamma\backslash\Gamma_0$, and then extending μ linearly to X. Then this μ is a linear functional on X, but it is not bounded. $\diamondsuit$

More useful than a Hamel basis is a *countable* sequence $\{x_n\}$ such that every element $x \in X$ can be written as some unique *infinite* linear combination $x = \sum c_n x_n$. This leads to the following definition.

Definition 4.2.

(a) A sequence $\{x_n\}$ in a Banach space X is a *basis* for X if

$$\forall\ x \in X, \quad \exists \text{ unique scalars } a_n(x) \text{ such that } x \ = \ \sum_n a_n(x)\, x_n. \tag{4.1}$$

(b) A basis $\{x_n\}$ is an *unconditional basis* if the series in (4.1) converges unconditionally for each $x \in X$.

(c) A basis $\{x_n\}$ is an *absolutely convergent basis* if the series in (4.1) converges absolutely for each $x \in X$.

(d) A basis $\{x_n\}$ is a *bounded basis* if $\{x_n\}$ is norm-bounded both above and below, i.e., if $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$.

(e) A basis $\{x_n\}$ is a *normalized basis* if $\{x_n\}$ is normalized, i.e, if $\|x_n\| = 1$ for every n. $\diamondsuit$

Absolutely convergent bases are studied in detail in Chapter 5. Unconditional bases are studied in detail in Chapter 9.

Note that if $\{x_n\}$ is a basis, then the fact that each $x \in X$ can be written uniquely as $x = \sum a_n(x)\, x_n$ implies that $x_n \neq 0$ for every n. As a consequence, $\{x_n/\|x_n\|\}$ is a normalized basis for X.

If X possesses a basis $\{x_n\}$ then X must be separable, since the set of all finite linear combinations $\sum_{n=1}^{N} c_n x_n$ with rational c_n (or rational real and imaginary parts if the c_n are complex) forms a countable, dense subset of X. The question of whether every separable Banach space possesses a basis was a longstanding problem known as the *Basis Problem.* It was shown by Enflo [Enf73] that there do exist separable, reflexive Banach spaces which do not possess any bases.

Notation 4.3. Note that the coefficients $a_n(x)$ defined in (4.1) are linear functions of x. Moreover, they are uniquely determined by the basis, i.e., the basis $\{x_n\}$ determines a unique collection of linear functionals $a_n \colon X \to \mathbf{F}$. We therefore call $\{a_n\}$ the *associated sequence of coefficient functionals.* Since these functionals are uniquely determined, we often do not declare them explicitly. When we do need to refer explicitly to both the basis and the associated coefficient functionals, we will write "$(\{x_n\}, \{a_n\})$ is a basis" to mean that $\{x_n\}$ is a basis with associated coefficient functionals $\{a_n\}$. We show in Theorem 4.11 that the coefficient functionals for any basis must be continuous, i.e., $\{a_n\} \subset X^*$.

Further, note that since $x_m = \sum a_n(x)\, x_n$ and $x_m = \sum \delta_{mn}\, x_n$ are two expansions of x_m, we must have $a_m(x_n) = \delta_{mn}$ for every m and n. We therefore say that the sequences $\{x_n\} \subset X$ and $\{a_n\} \subset X^*$ are *biorthogonal*, and we often say that $\{a_n\}$ is the *biorthogonal system* associated with $\{x_n\}$. General biorthogonal systems are considered in more detail in Chapter 7. In particular, we show there that the fact that $\{x_n\}$ is a basis implies that $\{a_n\}$ is the unique sequence in X^* that is biorthogonal to $\{x_n\}$. ♢

Example 4.4. Fix $1 \le p < \infty$, and consider the space $X = \ell^p$ defined in Example 1.6. Define sequences $e_n = (\delta_{mn})_{m=1}^{\infty} = (0, \dots, 0, 1, 0, \dots)$, where the 1 is in the nth position. Then $\{e_n\}$ is a basis for ℓ^p, often called the *standard basis* for ℓ^p. Note that $\{e_n\}$ is its own sequence of coefficient functionals.

On the other hand, $\{e_n\}$ is not a basis for ℓ^∞, and indeed ℓ^∞ has no bases whatsoever since it is not separable. Using the ℓ^∞ norm, the sequence $\{e_n\}$ is a basis for the space c_0 defined in Example 1.6(c). ♢

We are primarily interested in bases for which the coefficient functionals $\{a_n\}$ are *continuous.* We therefore give such bases a special name.

Definition 4.5. A basis $(\{x_n\}, \{a_n\})$ is a *Schauder basis* if each coefficient functional a_n is continuous. In this case, each a_n is an element of the dual space, i.e., $a_n \in X^*$ for every n. ♢

We shall see in Theorem 4.11 that *every* basis is a Schauder basis, i.e., the coefficient functionals a_n are *always* continuous. First, however, we require some definitions and miscellaneous facts. In particular, the following operators play a key role in analyzing bases.

Notation 4.6. The *partial sum operators*, or the *natural projections*, associated with the basis $(\{x_n\}, \{a_n\})$ are the mappings $S_N \colon X \to X$ defined by

$$S_N x \;=\; \sum_{n=1}^{N} a_n(x)\, x_n. \quad ♢$$

The partial sum operators are clearly linear. We will show in Corollary 4.8 that if $\{x_n\}$ is a basis then each partial sum operator S_N is a bounded mapping of X into itself. Then the fact that all bases are Schauder bases will follow from the continuity of the partial sum operators (Theorem 4.11). The next proposition will be a key tool in this analysis. It states that if $\{x_n\}$ is a basis, then it is possible to endow the space Y of all sequences (c_n) such that $\sum c_n x_n$ converges with a norm so that it becomes a Banach space isomorphic to X. In general, however, it is difficult or impossible to explicitly describe the space Y. One exception was discussed in Example 2.5: if $\{e_n\}$ is an orthonormal basis for a Hilbert space H, then $\sum c_n e_n$ converges if and only if $(c_n) \in \ell^2$.

Recall that a *topological isomorphism* between Banach spaces X and Y is a linear bijection $S\colon X \to Y$ that is continuous. By the Inverse Mapping Theorem (Theorem 1.44), every topological isomorphism has a continuous inverse $S^{-1}\colon Y \to X$.

Proposition 4.7. [Sin70, p. 18]. *Let $\{x_n\}$ be a sequence in a Banach space X, and assume that $x_n \neq 0$ for every n. Define $Y = \big\{(c_n) : \sum c_n x_n \text{ converges in } X\big\}$, and set*

$$\|(c_n)\|_Y \;=\; \sup_N \Big\| \sum_{n=1}^N c_n x_n \Big\|.$$

Then the following statements hold.

(a) *Y is a Banach space.*

(b) *If $\{x_n\}$ is a basis for X then Y is topologically isomorphic to X via the mapping $(c_n) \mapsto \sum c_n x_n$.*

Proof. (a) It is clear that Y is a linear space. If $(c_n) \in Y$ then $\sum c_n x_n = \lim_{N\to\infty} \sum_{n=1}^N c_n x_n$ converges. Since convergent sequences are bounded, we therefore have $\|(c_n)\|_Y < \infty$ for each $(c_n) \in Y$. Thus $\|\cdot\|_Y$ is well-defined. It is easy to see that $\|(c_n) + (d_n)\|_Y \le \|(c_n)\|_Y + \|(d_n)\|_Y$ and $\|a\,(c_n)\|_Y = |a|\,\|(c_n)\|_Y$ for every scalar a, so $\|\cdot\|_Y$ is at least a seminorm on Y. Suppose that $\|(c_n)\|_Y = 0$. Then $\|\sum_{n=1}^N c_n x_n\| = 0$ for every N. In particular, $\|c_1 x_1\| = 0$, so we must have $c_1 = 0$ since we have assumed $x_1 \neq 0$, But then $\|c_2 x_2\| = \|\sum_{n=1}^2 c_n x_n\| = 0$, so $c_2 = 0$, etc. Hence $\|\cdot\|_Y$ is a norm on Y.

It remains only to show that Y is complete in this norm. Let $A_N = (c_n^N)$ be any collection of sequences from Y which form a Cauchy sequence with respect to the norm $\|\cdot\|_Y$. Then for n fixed, we have

$$|c_n^M - c_n^N|\,\|x_n\| \;=\; \|(c_n^M - c_n^N)\,x_n\| \;\le\; \Big\|\sum_{k=1}^n (c_k^M - c_k^N)\,x_k\Big\| + \Big\|\sum_{k=1}^{n-1} (c_k^M - c_k^N)\,x_k\Big\| \;\le\; 2\,\|A_M - A_N\|_Y.$$

Since $\{A_N\}$ is Cauchy and $x_n \neq 0$, we conclude that $(c_n^N)_{N=1}^\infty$ is a Cauchy sequence of scalars, so must converge to some scalar c_n as $N \to \infty$.

Choose now any $\varepsilon > 0$. Then since $\{A_N\}$ is Cauchy in Y, there exists an integer $N_0 > 0$ such that

$$\forall\, M, N \ge N_0, \quad \|A_M - A_N\|_Y \;=\; \sup_L \Big\|\sum_{n=1}^L (c_n^M - c_n^N)\,x_n\Big\| \;<\; \varepsilon. \tag{4.2}$$

Fix $N \geq N_0$ and any $L > 0$, and set $y_M = \sum_{n=1}^{L}(c_n^M - c_n^N)\,x_n$. Then $\|y_M\| < \varepsilon$ for each $M \geq N_0$ by (4.2). However, $y_M \to y = \sum_{n=1}^{L}(c_n - c_n^N)\,x_n$, so we must have $\|y\| \leq \varepsilon$. Thus, we have shown that

$$\forall\, N \geq N_0, \quad \sup_L \left\| \sum_{n=1}^{L} (c_n - c_n^N)\,x_n \right\| \;\leq\; \varepsilon. \tag{4.3}$$

Further, $(c_n^{N_0})_{n=1}^{\infty} \in Y$, so $\sum_n c_n^{N_0}\,x_n$ converges by definition. Hence, there is an $M_0 > 0$ such that

$$\forall\, N > M \geq M_0, \quad \left\| \sum_{n=M+1}^{N} c_n^{N_0}\,x_n \right\| \;<\; \varepsilon.$$

Therefore, if $N > M \geq M_0,\ N_0$ then

$$\begin{aligned}
\left\| \sum_{n=M+1}^{N} c_n x_n \right\| \;&=\; \left\| \sum_{n=1}^{N} (c_n - c_n^{N_0})\,x_n \;-\; \sum_{n=1}^{M} (c_n - c_n^{N_0})\,x_n \;+\; \sum_{n=M+1}^{N} c_n^{N_0}\,x_n \right\| \\
&\leq\; \left\| \sum_{n=1}^{N} (c_n - c_n^{N_0})\,x_n \right\| \;+\; \left\| \sum_{n=1}^{M} (c_n - c_n^{N_0})\,x_n \right\| \;+\; \left\| \sum_{n=M+1}^{N} c_n^{N_0}\,x_n \right\| \\
&\leq\; \varepsilon \;+\; \varepsilon \;+\; \varepsilon \;=\; 3\varepsilon.
\end{aligned}$$

Therefore $\sum c_n x_n$ converges in X, so $A = (c_n) \in Y$. Finally, by (4.3), we know that $A_N \to A$ in the norm of Y, so Y is complete.

(b) Suppose now that $\{x_n\}$ is a basis for X. Define the map $T\colon Y \to X$ by $T(c_n) = \sum c_n x_n$. This mapping is well-defined by the definition of Y. It is clearly linear, and it is bijective because $\{x_n\}$ is a basis. Finally, if $(c_n) \in Y$ then

$$\|T(c_n)\| \;=\; \left\| \sum_{n=1}^{\infty} c_n x_n \right\| \;=\; \lim_{N\to\infty} \left\| \sum_{n=1}^{N} c_n x_n \right\| \;\leq\; \sup_N \left\| \sum_{n=1}^{N} c_n x_n \right\| \;=\; \|(c_n)\|_Y.$$

Therefore T is bounded, hence is a topological isomorphism of Y onto X. □

An immediate consequence of Proposition 4.7 is that the partial sum operators S_N are bounded.

Corollary 4.8. *Let $(\{x_n\}, \{a_n\})$ be a basis for a Banach space X. Then:*

(a) $\sup \|S_N x\| < \infty$ *for each* $x \in X$,

(b) $C = \sup \|S_N\| < \infty$, *and*

(c) $|||x||| = \sup \|S_N x\|$ *forms a norm on X equivalent to the initial norm $\|\cdot\|$ for X, and satisfies* $\|\cdot\| \leq |||\cdot||| \leq C\,\|\cdot\|$.

Proof. (a) Let Y be as in Proposition 4.7. Then $T\colon X \to Y$ defined by $T(c_n) = \sum c_n x_n$ is a topological isomorphism of X onto Y. Suppose that $x \in X$. Then we have by definition that $x = \sum a_n(x)\,x_n$ and that the scalars $a_n(x)$ are unique, so we must have $T^{-1}x = (a_n(x))$. Hence

$$\sup_N \|S_N x\| \;=\; \sup_N \left\| \sum_{n=1}^{N} a_n(x)\,x_n \right\| \;=\; \big\|(a_n(x))\big\|_Y \;=\; \|T^{-1}x\|_Y \;\leq\; \|T^{-1}\|\,\|x\| \;<\; \infty. \tag{4.4}$$

(b) From (4.4), we see that $\sup \|S_N\| \leq \|T^{-1}\| < \infty$.

(c) It is easy to see that $|||\cdot|||$ has the properties of at least a seminorm. Now, given $x \in X$ we have

$$|||x||| \;=\; \sup_N \|S_N x\| \;\le\; \sup_N \|S_N\|\,\|x\| \;=\; C\,\|x\|$$

and

$$\|x\| \;=\; \lim_{N\to\infty} \|S_N x\| \;\le\; \sup_N \|S_N x\| \;=\; |||x|||.$$

It follows from these two statements that $|||\cdot|||$ is in fact a norm, and is equivalent to $\|\cdot\|$. □

The number C appearing in Corollary 4.8 is important enough to be dignified with a name of its own.

Definition 4.9. If $(\{x_n\},\{a_n\})$ is a basis for a Banach space X, then its *basis constant* is the finite number $C = \sup \|S_N\|$. The basis constant satisfies $C \ge 1$. If the basis constant is $C = 1$, then the basis is said to be *monotone*. ♢

The basis constant does depend on the norm. Unless otherwise specified, the basis constant is always taken with respect to the original norm on X. Changing to an equivalent norm for X will not change the fact that $\{x_n\}$ is a basis, but it can change the basis constant for $\{x_n\}$. For example, we show now that the basis constant in the norm $|||\cdot|||$ is always 1.

Proposition 4.10. *Every basis is monotone with respect to the equivalent norm $|||\cdot|||$ defined in* Corollary 4.8(c).

Proof. Note first that the composition of the partial sum operators S_M and S_N satisfies the rule

$$S_M\,S_N \;=\; \begin{cases} S_M, & \text{if } M \le N,\\ S_N, & \text{if } M \ge N.\end{cases}$$

Therefore,

$$|||S_N x||| \;=\; \sup_M \|S_M S_N x\| \;=\; \sup\,\{\|S_1 x\|,\dots,\|S_N x\|\}.$$

Hence,

$$\sup_N |||S_N x||| \;=\; \sup_N \|S_N x\| \;=\; |||x|||.$$

It follows from this that $\sup |||S_N||| = 1$. □

Now we can prove our main result: the coefficient functionals for *every* basis are continuous!

Theorem 4.11. [Sin70, p. 20]. *Every basis $(\{x_n\},\{a_n\})$ for a Banach space X is a Schauder basis for X. In fact, the coefficients functionals a_n are continuous linear functionals on X which satisfy*

$$1 \;\le\; \|a_n\|\,\|x_n\| \;\le\; 2C, \tag{4.5}$$

where C is the basis constant for $(\{x_n\},\{a_n\})$.

Proof. Since each a_n is a linear functional on X, we need only show that each a_n is bounded and that (4.5) is satisfied. Given $x \in X$, we compute

$$\begin{aligned} |a_n(x)|\,\|x_n\| \;=\; \|a_n(x)\,x_n\| \;&=\; \left\|\sum_{n=1}^{n} a_k(x)\,x_k \;-\; \sum_{n=1}^{n-1} a_k(x)\,x_k\right\| \\ &\le\; \left\|\sum_{n=1}^{n} a_k(x)\,x_k\right\| \;+\; \left\|\sum_{n=1}^{n-1} a_k(x)\,x_k\right\| \\ &=\; \|S_n x\| \;+\; \|S_{n-1}x\| \\ &\le\; 2C\,\|x\|. \end{aligned}$$

Since each x_n is nonzero, we conclude that $\|a_n\| \le 2C/\|x_n\| < \infty$. The final inequality follows from computing $1 = a_n(x_n) \le \|a_n\|\,\|x_n\|$. □

Since the coefficient functionals a_n are therefore elements of X^*, we use the notations $a_n(x) = \langle x, a_n\rangle$ interchangeably. In fact, from this point onward our preferred notation is $\langle x, a_n\rangle$.

We end this chapter with several useful results concerning the invariance of bases under topological isomorphisms.

Lemma 4.12. [You80, p. 30]. *Bases are preserved by topological isomorphisms. That is, if $\{x_n\}$ is a basis for a Banach space X and $S\colon X \to Y$ is a topological isomorphism, then $\{Sx_n\}$ is a basis for Y.*

Proof. If y is any element of Y then $S^{-1}y \in X$, so there are unique scalars (c_n) such that $S^{-1}y = \sum c_n x_n$. Since S is continuous, this implies $y = S(S^{-1}y) = \sum c_n S x_n$. Suppose that $y = \sum b_n S x_n$ was another representation of y. Then the fact that S^{-1} is also continuous implies that $S^{-1}y = \sum b_n x_n$, and hence that $b_n = c_n$ for each n. Thus $\{Sx_n\}$ is a basis for Y. □

This lemma motivates the following definition.

Definition 4.13. Let X and Y be Banach spaces. A basis $\{x_n\}$ for X is *equivalent* to a basis $\{y_n\}$ for Y if there exists a topological isomorphism $S\colon X \to Y$ such that $Sx_n = y_n$ for all n. If $X = Y$ then we write $\{x_n\} \sim \{y_n\}$ to mean that $\{x_n\}$ and $\{y_n\}$ are equivalent bases for X. ♢

It is clear that $\sim$ is an equivalence relation on the set of all bases of a Banach space X.

Note that we could define, more generally, that a basis $\{x_n\}$ for X is equivalent to a *sequence* $\{y_n\}$ in Y if there exists a topological isomorphism $S\colon X \to Y$ such that $Sx_n = y_n$. However, by Lemma 4.12, it follows immediately that such a sequence must be a basis for Y.

Pelczynski and Singer showed in 1964 that there exist uncountably many nonequivalent normalized conditional bases in every infinite dimensional Banach space which has a basis.

We show below in Corollary 4.15 that all orthonormal bases in a Hilbert space are equivalent. More generally, we show in Chapter 11 that all bounded unconditional bases in a Hilbert space are equivalent (and hence must be equivalent to orthonormal bases). Lindenstrauss and Pelczynski showed in 1968 that a non-Hilbert space H in which all bounded unconditional bases are equivalent must be isomorphic either to the sequence space c_0 or to the sequence space ℓ^1.

We can now give a characterization of equivalent bases.

Theorem 4.14. [You80, p. 30]. *Let X and Y be Banach spaces. Let $\{x_n\}$ be a basis for X and let $\{y_n\}$ be a basis for Y. Then the following two statements are equivalent.*

(a) *$\{x_n\}$ is equivalent to $\{y_n\}$.*

(b) *$\sum c_n x_n$ converges in X if and only if $\sum c_n y_n$ converges in Y.*

Proof. (a) $\Rightarrow$ (b). Suppose that $\{x_n\}$ is equivalent to $\{y_n\}$. Then there is a topological isomorphism $S\colon X \to Y$ such that $Sx_n = y_n$ for every n. Since S is continuous, the convergence of $\sum c_n x_n$ in X therefore implies the convergence of $\sum c_n Sx_n$ in Y. Similarly, S^{-1} is continuous, so the convergence of $\sum c_n y_n$ in Y implies the convergence of $\sum c_n S^{-1} y_n$ in X. Therefore (b) holds.

(b) $\Rightarrow$ (a). Suppose that (b) holds. Let $\{a_n\} \subset X^*$ be the coefficient functionals for the basis $\{x_n\}$, and let $\{b_n\} \subset Y^*$ be the coefficient functionals for the basis $\{y_n\}$. Suppose that $x \in X$ is given. Then $x = \sum \langle x, a_n \rangle x_n$ converges in X, so $Sx = \sum \langle x, a_n \rangle y_n$ converges in Y. Clearly S defined in this way is linear. The fact that the expansion $x = \sum \langle x, a_n \rangle x_n$ is unique ensures that S is well-defined. Further, if $Sx = 0$ then $\sum 0\, y_n = 0 = Sx = \sum \langle x, a_n \rangle y_n$, and therefore $\langle x, a_n \rangle = 0$ for every n since $\{y_n\}$ is a basis. This implies $x = \sum \langle x, a_n \rangle x_n = 0$, so we conclude that S is injective. Next, if y is any element of y, then $y = \sum \langle y, b_n \rangle y_n$ converges in Y, so $x = \sum \langle y, b_n \rangle x_n$ converges in X. Since $x = \sum \langle x, a_n \rangle x_n$ and $\{x_n\}$ is a basis, this forces $\langle y, b_n \rangle = \langle x, a_n \rangle$ for every n. Hence $Sx = y$ and therefore S is surjective. Thus S is a bijection of X onto Y.

It remains only to show that S is continuous. For each N, define $T_N\colon X \to Y$ by $T_N x = \sum_{n=1}^N \langle x, a_n \rangle y_n$. Since each functional a_n is continuous, we conclude that each T_N is continuous. In fact,

$$\|T_N x\| \;=\; \left\|\sum_{n=1}^N \langle x, a_n \rangle y_n\right\| \;\le\; \sum_{n=1}^N |\langle x, a_n \rangle|\, \|y_n\| \;\le\; \|x\| \sum_{n=1}^N \|a_n\|\, \|y_n\|.$$

Since $T_N x \to Sx$, we conclude that $\|Sx\| \le \sup \|T_N x\| < \infty$ for each individual $x \in X$. By the Uniform Boundedness Principle (Theorem 1.42), it follows that $\sup \|T_N\| < \infty$. However, $\|S\| \le \sup \|T_N\|$, so S is a bounded mapping. □

Corollary 4.15. *All orthonormal bases in a Hilbert space are equivalent.*

Proof. Suppose that $\{e_n\}$ and $\{f_n\}$ are both orthonormal bases for a Hilbert space H. Then, by Theorem 1.19(a),

$$\sum_n c_n e_n \text{ converges} \quad\Longleftrightarrow\quad \sum_n |c_n|^2 < \infty \quad\Longleftrightarrow\quad \sum_n c_n f_n \text{ converges}.$$

Hence $\{e_n\} \sim \{f_n\}$ by Theorem 4.14. □

5. ABSOLUTELY CONVERGENT BASES IN BANACH SPACES

It is often desirable to have a basis $\{x_n\}$ such that the series $x = \sum \langle x, a_n \rangle x_n$ has some special convergence properties. In this section we study those bases which have the property that this series is always absolutely convergent. We will see that this is a highly restrictive condition, which implies that X is isomorphic to ℓ^1. In Chapter 9 we will study those bases for which the series $x = \sum \langle x, a_n \rangle x_n$ is always unconditionally convergent.

Definition 5.1. A basis $(\{x_n\}, \{a_n\})$ for a Banach space X is *absolutely convergent* if the series $x = \sum \langle x, a_n \rangle x_n$ converges absolutely in X for each $x \in X$. That is, we require that

$$\forall\, x \in X, \quad \sum_n |\langle x, a_n \rangle|\, \|x_n\| < \infty. \quad \diamondsuit$$

Theorem 5.2. [Mar69, p. 42]. *If a Banach space X possesses an absolutely convergent basis then X is topologically isomorphic to ℓ^1.*

Proof. Suppose that $(\{x_n\}, \{a_n\})$ is an absolutely convergent basis for X. Define the mapping $T \colon X \to \ell^1$ by $Tx = (\langle x, a_n \rangle \|x_n\|)$. Certainly T is a well-defined, injective, and linear map. Suppose that $y_N \in X$, that $y_N \to y \in X$, and that $Ty_N \to (c_n) \in \ell^1$. Then

$$\lim_{N\to\infty} \sum_n \Big| \langle y_N, a_n \rangle \|x_n\| \;-\; c_n \Big| \;=\; \lim_{N\to\infty} \|Ty_N - (c_n)\|_{\ell^1} \;=\; 0. \tag{5.1}$$

Since the coefficient functionals a_n are continuous, we have by (5.1) that

$$\langle y, a_n \rangle \|x_n\| \;=\; \lim_{N\to\infty} \langle y_N, a_n \rangle \|x_n\| \;=\; c_n.$$

Therefore $Ty = (c_n)$, so T is a closed mapping. We conclude from the Closed Graph Theorem (Theorem 1.46) that T is continuous.

Now choose any $(c_n) \in \ell^1$. Then $(\|c_n x_n\| / \|x_n\|) \in \ell^1$, so $x = \sum \frac{c_n}{\|x_n\|} x_n \in X$. However, $Tx = (c_n)$, so T is surjective. Therefore T is a topological isomorphism of X onto ℓ^1. In particular, it follows from the Inverse Mapping Theorem (Theorem 1.44) that T^{-1} is continuous. Alternatively, we can see this directly from the calculation

$$\|x\| \;=\; \Big\| \sum_n \langle x, a_n \rangle x_n \Big\| \;\le\; \sum_n |\langle x, a_n \rangle|\, \|x_n\| \;=\; \|Tx\|_{\ell^1}. \quad \square$$

Example 5.3. Let H be a separable, infinite-dimensional Hilbert space, and let $\{e_n\}$ be any orthonormal basis for H. We saw in Example 2.5 that $\sum c_n e_n$ converges if and only if $(c_n) \in \ell^2$, and that in this case the convergence is unconditional. On the other hand, since $\|e_n\| = 1$, we see that $\sum c_n e_n$ converges absolutely if and only if $(c_n) \in \ell^1$. Since ℓ^1 is a proper subset of ℓ^2, this implies that $\{e_n\}$ is not an absolutely convergent basis for H. Moreover, since H is topologically isomorphic to ℓ^2, and since ℓ^2 is not topologically isomorphic to ℓ^1, it follows from Theorem 5.2 that H does not possess any absolutely convergent bases. $\diamondsuit$

6. SOME TYPES OF LINEAR INDEPENDENCE OF SEQUENCES

In an infinite-dimensional Banach space, there are several possible types of linear independence of sequences. We list three of these in the following definition. We will consider minimal sequences in particular in more detail in Chapter 7.

Definition 6.1. A sequence $\{x_n\}$ in a Banach space X is:

(a) *finitely independent* if $\sum_{n=1}^{N} c_n x_n = 0$ implies $c_1 = \cdots = c_N = 0$,

(b) *ω-independent* if $\sum_{n=1}^{\infty} c_n x_n$ converges and equals 0 only when $c_n = 0$ for every n,

(c) *minimal* if $x_m \notin \overline{\text{span}}\{x_n\}_{n \neq m}$ for every m. $\diamondsuit$

Theorem 6.2. *Let $\{x_n\}$ be a sequence in a Banach space X. Then:*

(a) *$\{x_n\}$ is a basis $\implies$ $\{x_n\}$ is minimal and complete.*

(b) *$\{x_n\}$ is minimal $\implies$ $\{x_n\}$ is ω-independent.*

(c) *$\{x_n\}$ is ω-independent $\implies$ $\{x_n\}$ is finitely independent.*

Proof. (a) Assume that $(\{x_n\}, \{a_n\})$ is a basis for a Banach space X. Then $\{x_n\}$ is certainly complete, so we need only show that it is minimal. Fix m, and define $E = \text{span}\{x_n\}_{n \neq m}$. Then, since $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have $\langle x, a_m \rangle = 0$ for every $x \in E$. Since a_m is continuous, this implies $\langle x, a_m \rangle = 0$ for every $x \in \bar{E} = \overline{\text{span}}\{x_n\}_{n \neq m}$. However, we know that $\langle x_m, a_m \rangle = 1$, so we conclude that $x_m \notin \bar{E}$. Hence $\{x_n\}$ is minimal.

(b) Suppose that $\{x_n\}$ is minimal and that $\sum c_n x_n$ converges and equals 0. Let m be such that $c_m \neq 0$. Then $x_m = -\frac{1}{c_m} \sum_{m \neq n} c_n x_n \in \overline{\text{span}}\{x_n\}_{n \neq m}$, a contradiction.

(c) Clear. $\square$

None of the implications in Theorem 6.2 are reversible, as the following examples show.

Example 6.3. [Sin70, p. 24]. *Minimal and complete $\not\Longrightarrow$ basis.*

Define $C(\mathbf{T}) = \{f \in C(\mathbf{R}) : f(t+1) = f(t)\}$, the space of all continuous, 1-periodic functions. Then $C(\mathbf{T})$ is a Banach space under the uniform norm $\|\cdot\|_{L^\infty}$. Consider the functions $e_n(t) = e^{2\pi i n t}$ for $n \in \mathbf{Z}$. Not only are these functions elements of $C(\mathbf{T})$, but they define continuous linear functionals on $C(\mathbf{T})$ via the inner product $\langle f, e_n \rangle = \int_0^1 f(t)\, e^{-2\pi i n t}\, dt$. Further, $\{e_n\}_{n \in \mathbf{Z}}$ is its own biorthogonal system since $\langle e_m, e_n \rangle = \delta_{mn}$. Lemma 7.2 below therefore implies that $\{e_n\}_{n \in \mathbf{Z}}$ is minimal in $C(\mathbf{T})$. The Weierstrass Approximation Theorem [Kat68, p. 15] states that if $f \in C(\mathbf{T})$ then $\big\|f - \sum_{n=-N}^{N} c_n e_n\big\|_{L^\infty} < \varepsilon$ for some scalars c_n. Hence $\text{span}\{e_n\}_{n \in \mathbf{Z}}$ is dense in $C(\mathbf{T})$, and therefore $\{e_n\}_{n \in \mathbf{Z}}$ is complete in $C(\mathbf{T})$. Alternatively, we can demonstrate the completeness as follows. Suppose that $f \in C(\mathbf{T})$ satisfies $\langle f, e_n \rangle = 0$ for every n. Since $C(\mathbf{T}) \subset L^2(\mathbf{T})$ and since $\{e_n\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $L^2(\mathbf{T})$, this implies that f is the zero function in the space

$L^2(\mathbf{T})$, hence is zero almost everywhere. Since f is continuous, it follows that $f(t) = 0$ for all t. Hence $\{e_n\}_{n\in\mathbf{Z}}$ is complete both $C(\mathbf{T})$ and $L^2(\mathbf{T})$ by Corollary 1.41.

Thus, $\{e_n\}_{n\in\mathbf{Z}}$ is both minimal and complete in $C(\mathbf{T})$. Further, if $f = \sum c_n e_n$ converges in $C(\mathbf{T})$, then it is easy to see from the orthonormality of the e_n that $c_n = \langle f, e_n\rangle$. However, it is known that there exist continuous functions $f \in C(\mathbf{T})$ whose Fourier series $f = \sum \langle f, e_n\rangle\, e_n$ do not converge uniformly [Kat68, p. 51]. Therefore, $\{e_n\}_{n\in\mathbf{Z}}$ cannot be a basis for $C(\mathbf{T})$. $\diamondsuit$

Example 6.4. [Sin70, p. 24]. *ω-independent $\not\Rightarrow$ minimal.*

Let X be a Banach space such that there exists a sequence $\{x_n\}$ that is both minimal and complete in X but is not a basis for X (for example, we could use $X = C(\mathbf{T})$ and $x_n(t) = e_n(t) = e^{2\pi int}$ as in Example 6.3). Since $\{x_n\}$ is minimal, it follows from Lemma 7.2 that there exists a sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$. Since $\{x_n\}$ is not a basis, there must exist some $y \in X$ such that the series $\sum \langle y, a_n\rangle\, x_n$ does not converge in X. Consider the sequence $\{y\} \cup \{x_n\}$. This new sequence is certainly complete, and since $y \in \overline{\text{span}}\{x_n\}$, it cannot be minimal. However, we will show that $\{y\} \cup \{x_n\}$ is ω-independent. Assume that $cy + \sum c_n x_n = 0$, i.e., the summation converges and equals zero. If $c \neq 0$ then we would have $y = -\frac{1}{c}\sum c_n x_n$. The biorthogonality of $\{x_n\}$ and $\{a_n\}$ then implies that $\langle y, a_n\rangle = -c_n/c$. But then $\sum \langle y, a_n\rangle\, x_n$ converges, which is a contradiction. Therefore, we must have $c = 0$, and therefore $\sum c_n x_n = 0$. However, $\{x_n\}$ is minimal, and therefore is ω-independent, so this implies that every c_n is zero. Thus $\{y\} \cup \{x_n\}$ is ω-independent and complete, but not minimal.

Alternatively, we can give a Hilbert space example of a complete ω-independent sequence that is not minimal [VD97]. Let $\{e_n\}$ be any orthonormal basis for any separable Hilbert space H, and define $f_1 = e_1$ and $f_n = e_1 + e_n/n$ for $n \geq 2$. Then $\{f_n\}$ is certainly complete since $\text{span}\{f_n\} = \text{span}\{e_n\}$. However, $\|f_1 - f_n\| = \|e_n/n\| = 1/n \to 0$. Therefore $f_1 \in \overline{\text{span}}\{f_n\}_{n\geq 2}$, so $\{f_n\}$ is not minimal. To see that $\{f_n\}$ is ω-independent, suppose that $\sum c_n f_n$ converges and equals zero. Then

$$\sum_{n=1}^{N} c_n f_n \;=\; \left(\sum_{n=1}^{N} c_n\right) e_1 \;+\; \sum_{n=2}^{N} c_n e_n \;\to\; 0.$$

Therefore,

$$\left\| \left(\sum_{n=1}^{N} c_n\right) e_1 \;+\; \sum_{n=2}^{N} c_n e_n \right\|^2 \;=\; \left|\sum_{n=1}^{N} c_n\right|^2 \;+\; \sum_{n=2}^{N} |c_n|^2 \;\to\; 0.$$

This implies immediately that $c_n = 0$ for each $n \geq 2$, and therefore $c_1 = 0$ as well. $\diamondsuit$

Example 6.5. [Sin70, p. 25]. *Finitely independent $\not\Rightarrow$ ω-independent.*

Let $(\{x_n\}, \{a_n\})$ be a basis for a Banach space X, and let $x \in X$ be any element such that $\langle x, a_n\rangle \neq 0$ for every n. For example, we could take $x = \sum \frac{x_n}{2^n\, \|x_n\|}$. Note that x cannot equal any x_n because $\langle x_n, a_m\rangle = 0$ when $m \neq n$. Consider then the new sequence $\{x\} \cup \{x_n\}$. This is certainly complete, and $-x + \sum \langle x, a_n\rangle\, x_n = 0$, so it is not ω-independent. However, we will show that it is finitely independent. Suppose that $cx + \sum_{n=1}^{N} c_n x_n = 0$. Substituting the fact that

$x = \sum \langle x, a_n \rangle \, x_n$, it follows that

$$\sum_{n=1}^{N} \bigl(c\langle x, a_n\rangle + c_n\bigr)\, x_n \;+\; \sum_{n=N+1}^{\infty} c\langle x, a_n\rangle\, x_n \;=\; 0.$$

However, $\{x_n\}$ is a basis, so this is only possible if $c\langle x, a_n\rangle + c_n = 0$ for $n = 1, \dots, N$ and $c\langle x, a_n\rangle = 0$ for $n > N$. Since no $\langle x, a_n\rangle$ is zero we therefore must have $c = 0$. But then $c_1 = \cdots = c_N = 0$, so $\{x\} \cup \{x_n\}$ is finitely independent. $\diamondsuit$

7. BIORTHOGONAL SYSTEMS IN BANACH SPACES

A basis $\{x_n\}$ and its associated coefficient functionals $\{a_n\}$ are an example of biorthogonal sequences. We study the properties of general biorthogonal systems in this chapter.

Definition 7.1. Given a Banach space X and given sequences $\{x_n\} \subset X$ and $\{a_n\} \subset X^*$, we say that $\{a_n\}$ is *biorthogonal* to $\{x_n\}$, or that $(\{x_n\}, \{a_n\})$ is a *biorthogonal system*, if $\langle x_m, a_n \rangle = \delta_{mn}$ for every m, n. We associate with each biorthogonal system $(\{x_n\}, \{a_n\})$ the *partial sum operators* $S_N \colon X \to X$ defined by

$$S_N x \;=\; \sum_{n=1}^{N} \langle x, a_n \rangle \, x_n. \quad \diamondsuit$$

We show now that the existence of sequence biorthogonal to $\{x_n\}$ is equivalent to the statement that $\{x_n\}$ is minimal.

Lemma 7.2. [You80, p. 28], [Sin70, p. 53]. *Let X be a Banach space, and let $\{x_n\} \subset X$. Then:*

(a) $\exists \; \{a_n\} \subset X^*$ *biorthogonal to* $\{x_n\} \iff \{x_n\}$ *is minimal.*

(b) $\exists$ *unique* $\{a_n\} \subset X^*$ *biorthogonal to* $\{x_n\} \iff \{x_n\}$ *is minimal and complete.*

Proof. (a) $\Rightarrow$. Suppose that $\{a_n\} \subset X^*$ is biorthogonal to $\{x_n\}$. Fix any m, and choose $z \in \operatorname{span}\{x_n\}_{n \neq m}$, say $z = \sum_{j=1}^{N} c_{n_j} x_{n_j}$. Then $\langle z, a_m \rangle = \sum_{j=1}^{N} c_{n_j} \langle x_{n_j}, a_m \rangle = 0$ since $x_{n_j} \neq x_m$ for all j. Since a_m is continuous, we then have $\langle z, a_m \rangle = 0$ for all $z \in \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. However $\langle x_m, a_m \rangle = 1$, so we must have $x_m \notin \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. Therefore $\{x_n\}$ is minimal.

$\Leftarrow$. Suppose that $\{x_n\}$ is minimal. Fix m, and define $E = \overline{\operatorname{span}}\{x_n\}_{n \neq m}$. This is a closed subspace of X which does not contain x_m. Therefore, by the Hahn–Banach Theorem (Corollary 1.40) there is a functional $a_m \in X^*$ such that

$$\langle x_m, a_m \rangle \;=\; 1 \qquad \text{and} \qquad \langle x, a_m \rangle \;=\; 0 \text{ for } x \in E.$$

Repeating this for all m we obtain a sequence $\{a_n\}$ that is biorthogonal to $\{x_n\}$.

(b) $\Rightarrow$. Suppose there is a unique sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$. We know that $\{x_n\}$ is minimal by part (a), so it remains only to show that $\{x_n\}$ is complete. Suppose that $x^* \in X^*$ is a continuous linear functional such that $\langle x_n, x^* \rangle = 0$ for every n. Then

$$\langle x_m, x^* + a_n \rangle \;=\; \langle x_m, x^* \rangle + \langle x_m, a_n \rangle \;=\; 0 + \delta_{mn} = \delta_{mn}.$$

Thus $\{x^* + a_n\}$ is also biorthogonal to $\{x_n\}$. By our uniqueness assumption, we must have $x^* = 0$. The Hahn–Banach Theorem (Corollary 1.41) therefore implies that $\overline{\operatorname{span}}\{x_n\} = X$, so $\{x_n\}$ is complete.

$\Leftarrow$. Suppose that $\{x_n\}$ is both minimal and complete. By part (a) we know that there exists at least one sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$, so we need only show that this sequence is unique. Suppose that $\{b_n\} \subset X^*$ is also biorthogonal to $\{x_n\}$. Then $\langle x_n, a_m - b_m \rangle = \delta_{mn} - \delta_{mn} = 0$ for every m and n. However, $\{x_n\}$ is complete, so the Hahn–Banach Theorem (Corollary 1.41) implies that $a_m - b_m = 0$ for every m. Thus $\{a_n\}$ is unique. $\square$

Next, we characterize the additional properties that a minimal sequence must possess in order to be a basis.

Theorem 7.3. [Sin70, p. 25]. *Let $\{x_n\}$ be a sequence in a Banach space X. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is a basis for X.*

(b) *There exists a biorthogonal sequence $\{a_n\} \subset X^*$ such that*

$$\forall\, x \in X, \quad x \;=\; \sum_n \langle x, a_n \rangle\, x_n.$$

(c) *$\{x_n\}$ is complete and there exists a biorthogonal sequence $\{a_n\} \subset X^*$ such that*

$$\forall\, x \in X, \quad \sup_N \|S_N x\| \;<\; \infty.$$

(d) *$\{x_n\}$ is complete and there exists a biorthogonal sequence $\{a_n\} \subset X^*$ such that*

$$\sup_N \|S_N\| \;<\; \infty.$$

Proof. (a) $\Rightarrow$ (b). Follows immediately from the definition of basis and the fact that every basis is a Schauder basis (Theorem 4.11).

(b) $\Rightarrow$ (a). Assume that statement (b) holds. We need only show that the representation $x = \sum \langle x, a_n \rangle\, x_n$ is unique. However, each a_m is continuous, so if $x = \sum c_n x_n$, then $\langle x, a_m \rangle = \sum c_n \langle x_n, a_m \rangle = \sum c_n\, \delta_{mn} = c_m$.

(b) $\Rightarrow$ (c). Assume that statement (b) holds. Then the fact that every x can be written $x = \sum \langle x, a_n \rangle\, x_n$ implies that $\mathrm{span}\{x_n\}$ is dense is X, hence that $\{x_n\}$ is complete. Further, it implies that $x = \lim_{N\to\infty} S_N x$, i.e., that the sequence $\{S_N x\}$ is convergent. Therefore statement (c) holds since all convergent sequences are bounded.

(c) $\Rightarrow$ (d). Each S_N is a bounded linear operator mapping X into itself. Therefore, this implication follows immediately from the Uniform Boundedness Principle (Theorem 1.42).

(d) $\Rightarrow$ (b). Assume that statement (d) holds, and choose any $x \in \mathrm{span}\{x_n\}$, say $x = \sum_{n=1}^M c_n x_n$. Then, since S_N is linear and $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have for each $N \geq M$ that

$$S_N x \;=\; S_N\biggl(\sum_{m=1}^M c_m x_m\biggr) \;=\; \sum_{m=1}^M c_m\, S_N x_m \;=\; \sum_{m=1}^M c_m \sum_{n=1}^N \langle x_m, a_n \rangle\, x_n \;=\; \sum_{m=1}^M c_m x_m \;=\; x.$$

Therefore, we trivially have $x = \lim_{N\to\infty} S_N x = \sum \langle x, a_n\rangle\, x_n$ when $x \in \operatorname{span}\{x_n\}$.

Now we will show that $x = \lim_{N\to\infty} S_N x$ for arbitrary $x \in X$. Let $C = \sup \|S_N\|$, and let x be an arbitrary element of X. Since $\{x_n\}$ is complete, $\operatorname{span}\{x_n\}$ is dense in X. Therefore, given $\varepsilon > 0$ we can find an element $y \in \operatorname{span}\{x_n\}$ with $\|x - y\| < \varepsilon/(1+C)$, say $y = \sum_{m=1}^{M} c_m x_m$. Then for $N \geq M$ we have

$$\begin{aligned}
\|x - S_N x\| &\leq \|x-y\| + \|y - S_N y\| + \|S_N y - S_N x\| \\
&\leq \|x-y\| + 0 + \|S_N\|\, \|x-y\| \\
&\leq (1+C)\, \|x-y\| \\
&< \varepsilon.
\end{aligned}$$

Thus $x = \lim_{N\to\infty} S_N x = \sum \langle x, a_n\rangle\, x_n$ for arbitrary $x \in X$, as desired. $\square$

The next two theorems give a characterization of minimal sequences and bases in terms of the size of *finite* linear combinations of the sequence elements.

Theorem 7.4. [Sin70, p. 54]. *Given a sequence $\{x_n\}$ in a Banach space X with all $x_n \neq 0$, the following two statements are equivalent.*

(a) $\{x_n\}$ *is minimal.*

(b) $\forall\, M,\ \exists\, C_M \geq 1$ *such that*

$$\forall\, N \geq M, \quad \forall\, c_0, \dots, c_N, \quad \left\|\sum_{n=1}^{M} c_n x_n\right\| \leq C_M \left\|\sum_{n=1}^{N} c_n x_n\right\|.$$

Proof. (a) $\Rightarrow$ (b). Assume that $\{x_n\}$ is minimal. Then there exists a sequence $\{a_n\} \subset X^*$ that is biorthogonal to $\{x_n\}$. Let $\{S_N\}$ be the partial sum operators associated with $(\{x_n\}, \{a_n\})$. Suppose that $N \geq M$, and that $c_0, \dots, c_N$ are any scalars. Then

$$\left\|\sum_{n=1}^{M} c_n x_n\right\| = \left\|S_M\Bigl(\sum_{n=1}^{N} c_n x_n\Bigr)\right\| \leq \|S_M\| \left\|\sum_{n=1}^{N} c_n x_n\right\|.$$

Therefore statement (b) follows with $C_M = \|S_M\|$.

(b) $\Rightarrow$ (a). Assume that statement (b) holds, and let $E = \operatorname{span}\{x_n\}$. Given $x = \sum_{n=1}^{N} c_n x_n \in E$ and $M \leq N$, we have

$$\begin{aligned}
|c_M|\, \|x_M\| = \|c_M x_M\| &\leq \left\|\sum_{n=1}^{M} c_n x_n\right\| + \left\|\sum_{n=1}^{M-1} c_n x_n\right\| \\
&\leq C_M \left\|\sum_{n=1}^{N} c_n x_n\right\| + C_{M-1} \left\|\sum_{n=1}^{N} c_n x_n\right\| = (C_M + C_{M-1})\, \|x\|.
\end{aligned}$$

As $x_M \neq 0$, we therefore have

$$|c_M| \leq \frac{(C_M + C_{M-1})\,\|x\|}{\|x_M\|}. \tag{7.1}$$

In particular, $x = 0$ implies $c_1 = \cdots = c_N = 0$. Thus $\{x_n\}$ is finitely linearly independent. Since E is the finite linear span of $\{x_n\}$, this implies that every element of E has a *unique* representation of the form $x = \sum_{n=1}^N c_n x_n$. As a consequence, we can define a scalar-valued mapping a_m on the set E by $a_m\big(\sum_{n=1}^N c_n x_n\big) = c_m$ (where we set $c_m = 0$ if $m > N$). By (7.1), we have $|a_m(x)| \leq (C_m + C_{m-1})\,\|x\|/\|x_m\|$ for every $x \in E$, so a_m is continuous on E. Since E is dense in X, the Hahn–Banach Theorem (Corollary 1.38) implies that there is a continuous extension of a_m to all of X. This extended a_m is therefore a continuous linear functional on X which is biorthogonal to $\{x_n\}$. Lemma 7.2 therefore implies that $\{x_n\}$ is minimal. □

For an arbitrary minimal sequence, the constants C_M is Theorem 7.4 need not be uniformly bounded. Compare this to the situation for bases given in the following result.

Theorem 7.5. [LT77, p. 2]. *Let $\{x_n\}$ be a sequence in a Banach space X. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is a basis for X.*

(b) *$\{x_n\}$ is complete, $x_n \neq 0$ for all n, and there exists $C \geq 1$ such that*

$$\forall\, N \geq M, \quad \forall\, c_1, \dots, c_N, \quad \Big\|\sum_{n=1}^M c_n x_n\Big\| \;\leq\; C\,\Big\|\sum_{n=1}^N c_n x_n\Big\|. \tag{7.2}$$

In this case, the best constant C in (7.2) is the basis constant $C = \sup \|S_N\|$.

Proof. (a) ⇒ (b). Suppose that $\{x_n\}$ is a basis for X, and let $C = \sup \|S_N\|$ be the basis constant. Then $\{x_n\}$ is complete and $x_n \neq 0$ for every n. Fix $N \geq M$, and choose any $c_1, \dots, c_N$. Then

$$\Big\|\sum_{n=1}^M c_n x_n\Big\| \;=\; \Big\|S_M\Big(\sum_{n=1}^N c_n x_n\Big)\Big\| \;\leq\; \|S_M\|\,\Big\|\sum_{n=1}^N c_n x_n\Big\| \;\leq\; C\,\Big\|\sum_{n=1}^N c_n x_n\Big\|.$$

(b) ⇒ (a). Suppose that statement (b) holds. It then follows from Theorem 7.4 that $\{x_n\}$ is minimal, so by Lemma 7.2 there exists a biorthogonal system $\{a_n\} \subset X^*$. Let S_N denote the partial sum operators associated with $(\{x_n\}, \{a_n\})$. Since $\{x_n\}$ is complete, it suffices by Theorem 7.3 to show that $\sup \|S_N\| < \infty$.

So, suppose that $x = \sum_{n=1}^M c_n x_n \in \operatorname{span}\{x_n\}$. Then:

$$N \leq M \implies \|S_N x\| \;=\; \Big\|\sum_{n=1}^N c_n x_n\Big\| \;\leq\; C\,\Big\|\sum_{n=1}^M c_n x_n\Big\| \;=\; C\,\|x\|,$$

$$N > M \implies \|S_N x\| \;=\; \Big\|\sum_{n=1}^M c_n x_n\Big\| \;=\; \|x\|.$$

As $C \geq 1$ we therefore have $\|S_N x\| \leq C\,\|x\|$ for all N whenever $x \in \operatorname{span}\{x_n\}$. However, each S_N is continuous and $\operatorname{span}\{x_n\}$ is dense in X, so this inequality must therefore hold for all $x \in X$. Thus $\sup \|S_N\| \leq C < \infty$, as desired. This inequality also shows that the smallest possible value for C in (7.2) is $C = \sup \|S_N\|$. □

The following result is an application of Theorem 7.5. Given a basis $\{x_n\}$ for a Banach space X, it is often useful to have some bound on how much the elements x_n can be perturbed so that the resulting sequence remains a basis for X, or at least a basis for its closed linear span. The following result is classical, and is a typical example of perturbation theorems that apply to general bases. For specific types of bases in specific Banach spaces, it is often possible to derive sharper results. For a survey of results on basis perturbations, we refer to [RH71].

Theorem 7.6. *Let $(\{x_n\}, \{a_n\})$ be a basis for a Banach space X, with basis constant C. If $\{y_n\} \subset X$ is such that*

$$R \;=\; \sum_n \|a_n\|\,\|x_n - y_n\| \;<\; 1,$$

then $\{y_n\}$ is a basis for $\overline{\operatorname{span}}\{y_n\}$, and has basis constant $C' \leq \frac{1+R}{1-R}\,C$. Moreover, in this case, the basis $\{x_n\}$ for X and the basis $\{y_n\}$ for $Y = \overline{\operatorname{span}}\{y_n\}$ are equivalent in the sense of Definition 4.13.

Proof. Note that, by definition, $\{y_n\}$ is complete in $Y = \overline{\operatorname{span}}\{y_n\}$. Further, if some $y_n = 0$ then we would have $R \geq \|a_n\|\,\|x_n\| \geq 1$ by (4.5), which contradicts the fact that $R < 1$. Therefore, each y_n must be nonzero. By Theorem 7.5, it therefore suffices to show that there exists a constant B such that

$$\forall\, N \geq M, \quad \forall\, c_1, \dots, c_N, \qquad \Bigl\|\sum_{n=1}^{M} c_n y_n\Bigr\| \;\leq\; B\,\Bigl\|\sum_{n=1}^{N} c_n y_n\Bigr\|. \tag{7.3}$$

Further, if (7.3) holds, then Theorem 7.5 also implies that the basis constant C' for $\{y_n\}$ satisfies $C' \leq B$.

So, assume that $N \geq M$ and that $c_1, \dots, c_N$ are given. Before showing the existence of the constant B, we will establish several useful inequalities. First, since $\{x_n\}$ and $\{a_n\}$ are biorthogonal, we have that

$$\forall\, K \geq m, \quad |c_m| \;=\; \Bigl|\Bigl\langle \sum_{n=1}^{K} c_n x_n,\; a_m \Bigr\rangle\Bigr| \;\leq\; \|a_m\|\,\Bigl\|\sum_{n=1}^{K} c_n x_n\Bigr\|.$$

Therefore, for each $K > 0$ we have

$$\begin{aligned}
\Bigl\|\sum_{m=1}^{K} c_m\,(x_m - y_m)\Bigr\| \;&\leq\; \sum_{m=1}^{K} |c_m|\,\|x_m - y_m\| \\
&\leq\; \sum_{m=1}^{K} \biggl(\|a_m\|\,\Bigl\|\sum_{n=1}^{K} c_n x_n\Bigr\|\biggr)\,\|x_m - y_m\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{n=1}^{K} c_n x_n \right\| \sum_{m=1}^{K} \|a_m\| \, \|x_m - y_m\| \\
&\le R \left\| \sum_{n=1}^{K} c_n x_n \right\|. \qquad (7.4)
\end{aligned}
$$

As a consequence,

$$
\left\| \sum_{n=1}^{M} c_n y_n \right\| \le \left\| \sum_{n=1}^{M} c_n x_n \right\| + \left\| \sum_{n=1}^{M} c_n (y_n - x_n) \right\| \le (1+R) \left\| \sum_{n=1}^{M} c_n x_n \right\|. \qquad (7.5)
$$

Further, (7.4) implies that

$$
\left\| \sum_{n=1}^{N} c_n x_n \right\| \le \left\| \sum_{n=1}^{N} c_n y_n \right\| + \left\| \sum_{n=1}^{N} c_n (x_n - y_n) \right\| \le \left\| \sum_{n=1}^{N} c_n y_n \right\| + R \left\| \sum_{n=1}^{N} c_n x_n \right\|.
$$

Therefore,

$$
\left\| \sum_{n=1}^{N} c_n y_n \right\| \ge (1-R) \left\| \sum_{n=1}^{N} c_n x_n \right\|. \qquad (7.6)
$$

Finally, since $\{x_n\}$ is a basis with basis constant C, Theorem 7.5 implies that

$$
\left\| \sum_{n=1}^{M} c_n x_n \right\| \le C \left\| \sum_{n=1}^{N} c_n x_n \right\|. \qquad (7.7)
$$

Combining (7.5), (7.6), and (7.7), we obtain

$$
\left\| \sum_{n=1}^{M} c_n y_n \right\| \le (1+R) \left\| \sum_{n=1}^{M} c_n x_n \right\| \le (1+R)C \left\| \sum_{n=1}^{N} c_n x_n \right\| \le \frac{1+R}{1-R} C \left\| \sum_{n=1}^{N} c_n y_n \right\|.
$$

Hence (7.3) holds with $B = \frac{1+R}{1-R} C$, and therefore $\{y_n\}$ is a basis for $\overline{\operatorname{span}}\{y_n\}$ with basis constant $C' \le B$.

Finally, calculations similar to (7.5) and (7.6) imply that

$$
(1-R) \left\| \sum_{n=M+1}^{N} c_n x_n \right\| \le \left\| \sum_{n=M+1}^{N} c_n y_n \right\| \le (1+R) \left\| \sum_{n=M+1}^{N} c_n x_n \right\|.
$$

Hence $\sum c_n x_n$ converges if and only if $\sum c_n y_n$ converges. It therefore follows from Theorem 4.14 that $\{x_n\}$ is equivalent to $\{y_n\}$. □