11. RIESZ BASES IN HILBERT SPACES

Let H be a Hilbert space. Recall from Definition 4.13 that a basis $\{x_n\}$ for H is *equivalent* to a basis $\{y_n\}$ for H if there exists a topological isomorphism $S\colon H \to H$ such that $Sx_n = y_n$ for all n. In this case, we write $\{x_n\} \sim \{y_n\}$. It is clear that $\sim$ is an equivalence relation on the set of all bases for H. In particular, we saw in Corollary 4.15 that all orthonormal bases in H are equivalent. We will show in this chapter that the class of all bases that are equivalent to orthonormal bases coincides with the class of all bounded unconditional bases for H, and we will discuss some of the properties of such bases.

Definition 11.1. A basis $\{x_n\}$ for a Hilbert space H is a *Riesz basis* for H if it is equivalent to some (and therefore every) orthonormal basis for H. $\diamondsuit$

Clearly, all Riesz bases are equivalent since all orthonormal bases are equivalent.

Remark 11.2. We show in Theorem 11.9 that bounded unconditional bases and Riesz bases are equivalent. Hence a bounded basis is a Riesz basis if and only if it is unconditional. It is very difficult to exhibit a bounded basis for a Hilbert space H that is not a Riesz basis for H. Babenko [Bab48] proved that if $0 < \alpha < 1/2$, then $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$ is a bounded basis for $L^2[0,1]$ that is not a Riesz basis. It is easy to see that $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$ is minimal in $L^2[0,1]$, since $\{|t|^{-\alpha} e^{2\pi int}\}_{n\in\mathbf{Z}}$ is contained in $L^2[0,1]$ and is biorthogonal to $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$. However, the proof that $\{|t|^\alpha e^{2\pi int}\}_{n\in\mathbf{Z}}$ is a conditional basis is difficult. $\diamondsuit$

As with bases or unconditional bases, we can show that Riesz bases are preserved by topological isomorphisms.

Lemma 11.3. *Riesz bases are preserved by topological isomorphisms. That is, if $\{x_n\}$ is a Riesz basis for a Hilbert space H and $S\colon H \to K$ is a topological isomorphism, then $\{Sx_n\}$ is a Riesz basis for K.*

Proof. Since H possesses a basis, it is separable. Therefore K, being isomorphic to H, is separable as well. By Theorem 1.21, all separable Hilbert spaces are *isometrically* isomorphic, so there exists an *isometry* Z that maps H onto K. Further, by definition of Riesz basis, there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$. Since Z is an isometric isomorphism, the sequence $\{Ze_n\}$ is an orthonormal basis for K. Hence, STZ^{-1} is a topological isomorphism of K onto itself which has the property that $STZ^{-1}(Ze_n) = STe_n = Sx_n$. Hence $\{Sx_n\}$ is equivalent to an orthonormal basis for K, so we conclude that $\{Sx_n\}$ is a Riesz basis for K. $\square$

This yields one half of our characterization of Riesz bases.

Corollary 11.4. *All Riesz bases are bounded unconditional bases.*

Proof. Let $\{x_n\}$ be a Riesz basis for a Hilbert space H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $S\colon H \to H$ such that $Se_n = x_n$ for every n. However, $\{e_n\}$ is a bounded unconditional basis, and bounded unconditional bases are preserved by topological isomorphisms by Lemma 9.3(b), so $\{x_n\}$ must be a bounded unconditional basis for H. □

Before presenting the converse to this result, we require some basic facts about Riesz bases.

Lemma 11.5. *Let $(\{x_n\}, \{a_n\})$ and $(\{y_n\}, \{b_n\})$ be bases for a Hilbert space H. If $\{x_n\} \sim \{y_n\}$, then $\{a_n\} \sim \{b_n\}$.*

Proof. By Corollary 8.3, $(\{a_n\}, \{x_n\})$ and $(\{b_n\}, \{y_n\})$ are both bases for H. Suppose now that $\{x_n\} \sim \{y_n\}$. Then there exists a topological isomorphism $S\colon H \to H$ such that $Sx_n = y_n$ for every n. The adjoint mapping S^* is also a topological isomorphism of H onto itself, and we have

$$\langle x_m, S^* b_n\rangle \;=\; \langle Sx_m, b_n\rangle \;=\; \langle y_m, b_n\rangle \;=\; \delta_{mn} \;=\; \langle x_m, a_n\rangle.$$

Since $\{x_n\}$ is complete, it follows that $S^* b_n = a_n$ for every n, and therefore $\{a_n\} \sim \{b_n\}$. □

We obtain as a corollary a characterization of Riesz bases as those bases which are equivalent to their own biorthogonal systems.

Corollary 11.6. *Let $(\{x_n\}, \{y_n\})$ be a basis for a Hilbert space H. Then the following statements are equivalent.*

(a) *$\{x_n\}$ is a Riesz basis for H.*

(b) *$\{y_n\}$ is a Riesz basis for H.*

(c) *$\{x_n\} \sim \{y_n\}$.*

Proof. (a) $\Rightarrow$ (b), (c). Assume that $\{x_n\}$ is a Riesz basis for H. Then $\{x_n\} \sim \{e_n\}$ for some orthonormal basis $\{e_n\}$ of H. By Lemma 11.5, it follows that $\{x_n\}$ and $\{e_n\}$ have equivalent biorthogonal systems. However, $\{e_n\}$ is biorthogonal to itself, so this implies $\{y_n\} \sim \{e_n\} \sim \{x_n\}$. Hence $\{y_n\}$ is equivalent to $\{x_n\}$, and $\{y_n\}$ is a Riesz basis for H.

(b) $\Rightarrow$ (a), (c). By Corollary 8.3, $(\{y_n\}, \{x_n\})$ is a basis for H. Therefore, this argument follows symmetrically.

(c) $\Rightarrow$ (a), (b). Assume that $\{x_n\} \sim \{y_n\}$. Then there exists a topological isomorphism $S\colon H \to H$ such that $Sx_n = y_n$ for every n. Since $(\{x_n\}, \{y_n\})$ is a basis, it follows that for each $x \in H$,

$$x \;=\; \sum_n \langle x, y_n\rangle\, x_n \;=\; \sum_n \langle x, Sx_n\rangle\, x_n,$$

whence

$$Sx \;=\; \sum_n \langle x, Sx_n\rangle\, Sx_n.$$

Therefore,

$$\langle Sx, x\rangle \;=\; \sum_n |\langle x, Sx_n\rangle|^2 \;\geq\; 0.$$

Thus S is a continuous and positive linear operator on H, and therefore has a continuous and positive square root $S^{1/2}$ [Wei80, Theorem 7.20]. Similarly, S^{-1} is positive and has a positive square root, which must be $S^{-1/2} = (S^{1/2})^{-1}$. Thus $S^{1/2}$ is a topological isomorphism of H onto itself. Moreover, $S^{1/2}$ is self-adjoint, so

$$\langle S^{1/2}x_m, S^{1/2}x_n\rangle \;=\; \langle x_m, S^{1/2}S^{1/2}x_n\rangle \;=\; \langle x_m, Sx_n\rangle \;=\; \langle x_m, y_n\rangle \;=\; \delta_{mn}.$$

Hence $\{S^{1/2}x_n\}$ is an orthonormal sequence in H, and it must be complete since $\{x_n\}$ is complete and $S^{1/2}$ is a topological isomorphism. Therefore $\{x_n\}$ is the image of the orthonormal basis $\{S^{1/2}x_n\}$ under the topological isomorphism $S^{-1/2}$. Hence $\{x_n\}$ is a Riesz basis. By symmetry, $\{y_n\}$ is a Riesz basis as well. □

Definition 11.7. A sequence $\{x_n\}$ in a Hilbert space H is a *Bessel sequence* if

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n\rangle|^2 \;<\; \infty. \quad \diamondsuit$$

Lemma 11.8. *If $\{x_n\}$ is a Bessel sequence, then the* coefficient mapping $Ux = (\langle x, x_n\rangle)$ *is a continuous linear mapping of H into ℓ^2. In other words, there exists a constant $B > 0$ such that*

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|.$$

Proof. We will use the Closed Graph Theorem (Theorem 1.46) to show that U is continuous. Suppose that $y_N \to y \in H$, and that $Uy_N \to (c_n) \in \ell^2$. Then for each fixed m,

$$\bigl|c_m - \langle y_N, x_m\rangle\bigr| \;\le\; \biggl(\sum_n \bigl|c_n - \langle y_N, x_n\rangle\bigr|\biggr)^{1/2} \;=\; \bigl\|(c_n) - Uy_N\bigr\|_{\ell^2} \;\to\; 0 \quad \text{as } N \to \infty.$$

Therefore $c_m = \lim_{N\to\infty} \langle y_N, x_m\rangle = \langle y, x_m\rangle$ for every m. Hence $(c_n) = (\langle y, x_m\rangle) = Uy$, so U has a closed graph, and therefore is continuous. □

The constant B in Lemma 11.8 is sometimes referred to as a *Bessel bound* or *upper frame bound* for $\{x_n\}$ (compare Definition 12.1).

Now we can prove that Riesz bases and bounded unconditional bases are equivalent.

Theorem 11.9. [GK69, p. 320], [You80, p. 32]. *If $\{x_n\}$ be a sequence in a Hilbert space H, then the following statements are equivalent.*

(a) *$\{x_n\}$ is a Riesz basis for H.*

(b) *$\{x_n\}$ is a bounded unconditional basis for H.*

(c) $\{x_n\}$ *is a basis for* H, *and*

$$\sum_n c_n x_n \text{ converges} \iff \sum_n |c_n|^2 < \infty.$$

(d) $\{x_n\}$ *is complete in* H *and there exist constants* A, $B > 0$ *such that*

$$\forall\, c_1, \dots, c_N, \quad A \sum_{n=1}^{N} |c_n|^2 \;\le\; \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2.$$

(e) *There is an equivalent inner product* $(\cdot,\cdot)$ *for* H *such that* $\{x_n\}$ *is an orthonormal basis for* H *with respect to* $(\cdot,\cdot)$.

(f) $\{x_n\}$ *is a complete Bessel sequence and possesses a biorthogonal system* $\{y_n\}$ *that is also a complete Bessel sequence.*

Proof. (a) $\Rightarrow$ (b). This is the content of Corollary 11.4.

(a) $\Leftrightarrow$ (c). Assume that $\{x_n\}$ is a basis for H, and let $\{e_n\}$ be any orthonormal basis for H. Then $\{x_n\}$ is a Riesz basis for H if and only if $\{x_n\} \sim \{e_n\}$. By Theorem 4.14, $\{x_n\} \sim \{e_n\}$ if and only if

$$\sum_n c_n x_n \text{ converges} \iff \sum_n c_n e_n \text{ converges.}$$

However, by Theorem 1.19(a),

$$\sum_n c_n e_n \text{ converges} \iff \sum_n |c_n|^2 < \infty.$$

Hence, statement (a) holds if and only if statement (d) holds.

(a) $\Rightarrow$ (d). Suppose that $\{x_n\}$ is a Riesz basis for H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $S\colon H \to H$ such that $Se_n = x_n$ for every n. Therefore, for any scalars $c_1, \dots, c_N$ we have

$$\Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|^2 \;=\; \Bigl\|S\Bigl(\sum_{n=1}^{N} c_n e_n\Bigr)\Bigr\|^2 \;\le\; \|S\|^2 \Bigl\|\sum_{n=1}^{N} c_n e_n\Bigr\|^2 \;=\; \|S\|^2 \sum_{n=1}^{N} |c_n|^2,$$

the last equality following from the Plancherel formula (Theorem 1.20). Similarly,

$$\sum_{n=1}^{N} |c_n|^2 \;=\; \Bigl\|\sum_{n=1}^{N} c_n e_n\Bigr\|^2 \;=\; \Bigl\|S^{-1}\Bigl(\sum_{n=1}^{N} c_n x_n\Bigr)\Bigr\|^2 \;\le\; \|S^{-1}\|^2 \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|^2.$$

Hence statement (d) holds with $A = \|S^{-1}\|^{-2}$ and $B = \|S\|^2$.

(a) ⇒ (e). Suppose that $\{x_n\}$ is a Riesz basis for H. Then there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $S\colon H \to H$ such that $Se_n = x_n$ for every n. Define

$$(x,y) \;=\; \langle Sx, Sy\rangle \qquad\text{and}\qquad |||x|||^2 \;=\; (x,x) \;=\; \langle Sx, Sx\rangle \;=\; \|Sx\|^2.$$

It is easy to see that $(\cdot,\cdot)$ is an inner product for H, and that $|||\cdot|||$ is the corresponding induced norm. Further,

$$|||x|||^2 \;=\; \|Sx\|^2 \;\le\; \|S\|^2\,\|x\|^2 \qquad\text{and}\qquad \|x\|^2 \;=\; |||S^{-1}x|||^2 \;\le\; |||S^{-1}|||^2\,|||x|||^2, \tag{11.1}$$

where $|||S^{-1}|||$ is the operator norm of S^{-1} with respect to the norm $|||\cdot|||$. In fact, we have

$$|||S^{-1}||| \;=\; \sup_{|||x|||=1} |||S^{-1}x||| \;=\; \sup_{\|Sx\|=1} \|x\| \;=\; \sup_{\|y\|=1} \|S^{-1}y\| \;=\; \|S^{-1}\|,$$

although this equality is not needed for our proof. It follows from (11.1) that $|||\cdot|||$ is an equivalent norm to $\|\cdot\|$. By definition, $(\cdot,\cdot)$ is therefore an equivalent inner product to $\langle\cdot,\cdot\rangle$.

It remains to show that $\{x_n\}$ is an orthonormal basis with respect to the inner product $(\cdot,\cdot)$. By Theorem 1.19, it suffices to show that $\{x_n\}$ is a complete orthonormal sequence with respect to $(\cdot,\cdot)$. The orthonormality follows from the calculation

$$(x_m, x_n) \;=\; \langle Sx_m, Sx_n\rangle \;=\; \langle e_m, e_n\rangle \;=\; \delta_{mn}.$$

For the completeness, suppose that there is an $x \in H$ such that $(x, x_n) = 0$ for every n. Then $0 = (x, x_n) = \langle Sx, Sx_n\rangle = \langle Sx, e_n\rangle$ for every n. Since $\{e_n\}$ is complete with respect to $\langle\cdot,\cdot\rangle$, this implies that $Sx = 0$. Since S is a topological isomorphism, we therefore have $x = 0$. Hence $\{x_n\}$ is complete with respect to $(\cdot,\cdot)$.

(a) ⇒ (f). Suppose that $\{x_n\}$ is a Riesz basis for H. Then, by Corollary 11.6, $\{x_n\}$ possesses a biorthogonal sequence $\{y_n\}$ which is itself a Riesz basis for H. Suppose now that $x \in H$. Then since $(\{x_n\},\{y_n\})$ is a basis, we have that $x = \sum \langle x, y_n\rangle\, x_n$. Since we have already shown that statement (a) implies statement (c), the convergence of this series implies that $\sum |\langle x, y_n\rangle|^2 < \infty$. Therefore $\{y_n\}$ is a Bessel sequence. Further, $\{y_n\}$ is complete since $(\{y_n\},\{x_n\})$ is also a basis for H (Corollary 8.3). A symmetric argument implies that $\{x_n\}$ is a complete Bessel sequence as well.

(b) ⇒ (f). Suppose that $(\{x_n\},\{y_n\})$ is a bounded unconditional basis for H. Then, by Corollary 8.3, $(\{y_n\},\{x_n\})$ is also a bounded unconditional basis for H. Therefore, if $x \in H$ then $x = \sum \langle x, x_n\rangle\, y_n$, with unconditional convergence of this series. By Orlicz's Theorem (Theorem 3.1), this implies that $\sum |\langle x, x_n\rangle|^2\, \|y_n\|^2 < \infty$. However, by definition of bounded basis, there exist constants C_1, C_2 so that $0 < C_1 \le \|y_n\| \le C_2 < \infty$ for all n. Hence $\sum |\langle x, x_n\rangle|^2 < \infty$, so $\{x_n\}$ is a Bessel sequence, and it must be complete since it is a basis. A symmetric argument implies that $\{y_n\}$ is also a complete Bessel sequence.

(d) ⇒ (a). Suppose that statement (d) holds, and let $\{e_n\}$ be any orthonormal basis for H. Choose any $x \in H$. Then, by Theorem 1.20, $x = \sum \langle x, e_n\rangle\, e_n$, and $\sum |\langle x, e_n\rangle|^2 = \|x\|^2 < \infty$. Choose $M < N$, and define $c_1 = \cdots = c_M = 0$ and $c_n = \langle x, e_n\rangle$ for $n = M+1, \dots, N$. Then, by hypothesis (d),

$$\Bigl\| \sum_{n=M+1}^{N} \langle x, e_n\rangle\, x_n \Bigr\|^2 \;=\; \Bigl\| \sum_{n=1}^{N} c_n x_n \Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |c_n|^2 \;=\; B \sum_{n=M+1}^{N} |\langle x, e_n\rangle|^2.$$

Since $\sum |\langle x, e_n\rangle|^2$ is a Cauchy series of real numbers, it follows that $\sum \langle x, e_n\rangle\, x_n$ is a Cauchy series in H and hence must converge in H. Therefore, we can define $Sx = \sum \langle x, e_n\rangle\, x_n$. Clearly S defined in this way is a linear mapping of H into itself, and we claim that S is a topological isomorphism of H onto itself.

By applying hypothesis (d) and taking the limit as $N \to \infty$, we have

$$A\,\|x\|^2 \;=\; A \sum_n |\langle x, e_n\rangle|^2 \;\le\; \|Sx\|^2 \;\le\; B \sum_n |\langle x, e_n\rangle|^2 \;=\; B\,\|x\|^2. \tag{11.2}$$

It follows that S is continuous and injective, and that $S^{-1}\colon \mathrm{Range}(S) \to H$ is continuous as well. Further, $Se_m = \sum \langle e_m, e_n\rangle\, x_n = x_n$ for every n, so $\mathrm{Range}(S)$ contains every x_m, and therefore contains $\mathrm{span}\{x_n\}$, which is dense in H since $\{x_n\}$ is complete. Therefore, if we show that $\mathrm{Range}(S)$ is closed, then it will follow that $\mathrm{Range}(S) = H$ and hence that S is a topological isomorphism of H onto itself.

Suppose then that $y_n \in \mathrm{Range}(S)$ and that $y_n \to y \in H$. Then there exist $x_n \in H$ such that $Sx_n = y_n$. Hence $\{Sx_n\}$ is a Cauchy sequence in H. However, by (11.2) we have $A\,\|x_m - x_n\| \le \|Sx_m - Sx_n\|$, so $\{x_n\}$ is Cauchy as well, and therefore must converge to some $x \in H$. Since S is continuous, it follows that $y_n = Sx_n \to Sx$. Since we also have $y_n \to y$, we must have $y = Sx \in \mathrm{Range}(S)$. Hence $\mathrm{Range}(S)$ is closed.

Thus S is a topological isomorphism of H onto itself. Finally, since S maps $\{e_n\}$ onto $\{x_n\}$, we conclude that $\{x_n\}$ is a Riesz basis for H.

(d) ⇒ (b). Suppose that statement (d) holds. Choose $N > 0$, and define $a_n = \delta_{Nn}$. Then, by hypothesis (d),

$$A \;=\; A \sum_{n=1}^{N} |a_n|^2 \;\le\; \Bigl\| \sum_{n=1}^{N} a_n x_n \Bigr\|^2 \;=\; \|x_N\|^2 \;=\; \Bigl\| \sum_{n=1}^{N} a_n x_n \Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |a_n|^2 \;=\; B.$$

Hence $\{x_n\}$ is norm-bounded above and below. In particular, each x_n is nonzero.

It remains to show that $\{x_n\}$ is an unconditional basis. Therefore, choose any scalars $c_1, \dots, c_N$ and any signs $\varepsilon_1, \dots, \varepsilon_N = \pm 1$. Then by hypothesis (d),

$$\Bigl\| \sum_{n=1}^{N} \varepsilon_n c_n x_n \Bigr\|^2 \;\le\; B \sum_{n=1}^{N} |\varepsilon_n c_n|^2 \;=\; B \sum_{n=1}^{N} |c_n|^2 \;\le\; \frac{B}{A} \Bigl\| \sum_{n=1}^{N} c_n x_n \Bigr\|^2.$$

This, combined with the fact that $\{x_n\}$ is complete and that every x_n is nonzero, implies by Theorem 9.7 that $\{x_n\}$ is an unconditional basis for H.

(d) ⇒ (c). Suppose that statement (d) holds. Choose $N > 0$, and define $a_n = \delta_{Nn}$. Then by hypothesis (d),

$$\|x_N\|^2 \;=\; \Bigl\|\sum_{n=1}^{N} a_n x_n\Bigr\|^2 \;\geq\; A \sum_{n=1}^{N} |a_n|^2 \;=\; A.$$

Hence $\{x_n\}$ is norm-bounded below. In particular, each x_n is nonzero.

We will show now that $\{x_n\}$ is a basis for H. To do this, choose any $M < N$, and any scalars $c_1, \dots, c_N$. Then, by hypothesis (d),

$$\Bigl\|\sum_{n=1}^{M} c_n x_n\Bigr\|^2 \;\leq\; B \sum_{n=1}^{M} |c_n|^2 \;\leq\; B \sum_{n=1}^{N} |c_n|^2 \;\leq\; \frac{B}{A} \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|^2.$$

This, combined with the fact that $\{x_n\}$ is complete and that every x_n is nonzero, implies by Theorem 7.3 that $\{x_n\}$ is a basis.

It therefore only remains to show that $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$. To do this, let (c_n) be any sequence of scalars. Choose any $M < N$, and define $a_1 = \cdots = a_M = 0$ and $a_n = c_n$ for $n = M+1, \dots, N$. Then, by hypothesis (d),

$$A \sum_{n=1}^{N} |a_n|^2 \;\leq\; \Bigl\|\sum_{n=1}^{N} a_n x_n\Bigr\|^2 \;\leq\; B \sum_{n=1}^{N} |a_n|^2.$$

However, by the definition of a_n, this simply states that

$$A \sum_{n=M+1}^{N} |c_n|^2 \;\leq\; \Bigl\|\sum_{n=M+1}^{N} c_n x_n\Bigr\|^2 \;\leq\; B \sum_{n=M+1}^{N} |c_n|^2.$$

Therefore, $\sum c_n x_n$ is a Cauchy series in H if and only if $\sum |c_n|^2$ is a Cauchy series of real numbers. Hence one series converges if and only if the other series converges.

(e) ⇒ (d). Suppose that $(\cdot,\cdot)$ is an equivalent inner product for H such that $\{x_n\}$ is an orthonormal basis with respect to $(\cdot,\cdot)$. Let $|\!|\!| \cdot |\!|\!|$ denote the norm induced by $(\cdot,\cdot)$. Then, by definition of equivalent inner product, $|\!|\!| \cdot |\!|\!|$ and $\|\cdot\|$ are equivalent norms, i.e., there exist constants A, $B > 0$ such that

$$\forall\, x \in H, \quad A\, |\!|\!| x |\!|\!|^2 \;\leq\; \|x\|^2 \leq B\, |\!|\!| x |\!|\!|^2. \tag{11.3}$$

Since $\{x_n\}$ is complete in the norm $|\!|\!| \cdot |\!|\!|$ and since $\|\cdot\|$ is equivalent to $|\!|\!| \cdot |\!|\!|$, we must have that $\{x_n\}$ is complete in H with respect to $\|\cdot\|$. To see this explicitly, suppose that $x \in H$ and that $\varepsilon > 0$ is given. Then since $\operatorname{span}\{x_n\}$ is dense in H with respect to the norm $|\!|\!| \cdot |\!|\!|$, there must exist $y \in \operatorname{span}\{x_n\}$ such that $|\!|\!| x - y |\!|\!| < \varepsilon$. By (11.3), we therefore have $\|x - y\| < B^{1/2}\varepsilon$. Hence $\operatorname{span}\{x_n\}$ is also dense in H with respect to $\|\cdot\|$, and therefore $\{x_n\}$ is complete with respect to this norm.

Now choose any scalars $c_1, \dots, c_N$. Since $\{x_n\}$ is orthonormal with respect to $(\cdot,\cdot)$, we have by the Plancherel formula (Theorem 1.20) that $|\!|\!| \sum_{n=1}^{N} c_n x_n |\!|\!|^2 = \sum_{n=1}^{N} |c_n|^2$. Combined with (11.3), this implies that

$$A \sum_{n=1}^{N} |c_n|^2 \;\leq\; \Bigl\|\sum_{n=1}^{N} c_n x_n\Bigr\|^2 \;\leq\; B \sum_{n=1}^{N} |c_n|^2.$$

Hence statement (d) holds.

(f) ⇒ (c). Suppose that statement (f) holds. Since $\{x_n\}$ and $\{y_n\}$ are both Bessel sequences, it follows from Lemma 11.8 that there exist constants C, $D > 0$ such that

$$\forall\, x \in H, \quad \sum_n |\langle x, x_n\rangle|^2 \;\le\; C\,\|x\|^2 \qquad \text{and} \qquad \sum_n |\langle x, y_n\rangle|^2 \;\le\; D\,\|x\|^2. \tag{11.4}$$

We will show now that $(\{x_n\},\{y_n\})$ is a basis for H. Since $\{x_n\}$ is assumed to be complete and since $\{y_n\}$ is biorthogonal to $\{x_n\}$, it suffices by Theorem 7.3 to show that $\sup \|S_N\| < \infty$, where S_N is the partial sum operator $S_N x = \sum_{n=1}^N \langle x, y_n\rangle\, x_n$. We compute:

$$\begin{aligned}
\|S_N x\|^2 &= \sup_{\|y\|=1} |\langle S_N x, y\rangle|^2 && \text{by Theorem 1.16(b)}\\
&= \sup_{\|y\|=1} \left|\sum_{n=1}^N \langle x, y_n\rangle\, \langle x_n, y\rangle\right|^2\\
&\le \sup_{\|y\|=1} \left(\sum_{n=1}^N |\langle x, y_n\rangle|^2\right)\left(\sum_{n=1}^N |\langle x_n, y\rangle|^2\right) && \text{by Cauchy–Schwarz}\\
&\le \sup_{\|y\|=1} D\,\|x\|^2\, C\,\|y\|^2 && \text{by (11.4)}\\
&= CD\,\|x\|^2.
\end{aligned}$$

Hence $\sup \|S_N\|^2 \le CD < \infty$, as desired.

Finally, we must show that $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$. Suppose first that $x = \sum c_n x_n$ converges. Then we must have $c_n = \langle x, y_n\rangle$ since $(\{x_n\},\{y_n\})$ is a basis for H. It therefore follows from (11.4) that $\sum |c_n|^2 = \sum |\langle x, y_n\rangle|^2 \le D\,\|x\|^2 < \infty$.

Conversely, suppose that $\sum |c_n|^2 < \infty$. Then for any $M < N$,

$$\begin{aligned}
\left\|\sum_{n=M+1}^N c_n x_n\right\|^2 &= \sup_{\|y\|=1} \left|\left\langle \sum_{n=M+1}^N c_n x_n,\; y\right\rangle\right|^2 && \text{by Theorem 1.16(b)}\\
&= \sup_{\|y\|=1} \left|\sum_{n=M+1}^N c_n\, \langle x_n, y\rangle\right|^2\\
&\le \sup_{\|y\|=1} \left(\sum_{n=M+1}^N |c_n|^2\right)\left(\sum_{n=M+1}^N |\langle x_n, y\rangle|^2\right) && \text{by Cauchy–Schwarz}\\
&\le \sup_{\|y\|=1} \left(\sum_{n=M+1}^N |c_n|^2\right) C\,\|y\|^2 && \text{by (11.4)}\\
&= C \sum_{n=M+1}^N |c_n|^2.
\end{aligned}$$

Hence $\sum c_n x_n$ is a Cauchy series in H, and therefore must converge. □