12. FRAMES IN HILBERT SPACES

Frames were introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series [DS52]. They were intended as an alternative to orthonormal or Riesz bases in Hilbert spaces. Much of the abstract theory of frames is elegantly laid out in that paper. Frames for $L^2(\mathbf{R})$ based on time-frequency or time-scale translates of functions were later constructed by Daubechies, Grossmann, and Meyer in [DGM86]. Such frames play an important role in Gabor and wavelet analysis. Expository discussions of these connections can be found in [Dau92] and [HW89]. Gröchenig has given the nontrivial extension of frames to Banach spaces [Grö91].

This chapter is an essentially expository review of basic results on frames in Hilbert spaces. We have combined results from many sources, including [Dau90], [DGM86], [DS52], [You80] and others, with remarks, examples, and minor results of our own. This chapter is based on [Hei90] and [HW89].

Definition 12.1. A sequence $\{x_n\}$ in a Hilbert space H is a *frame* for H if there exist constants A, $B > 0$ such that the following *pseudo–Plancherel formula* holds:

$$\forall\, x \in H, \quad A\,\|x\|^2 \;\le\; \sum_n |\langle x, x_n\rangle|^2 \;\le\; B\,\|x\|^2. \tag{12.1}$$

The constants A, B are *frame bounds*; A is the *lower* bound and B is the *upper* bound. The frame is *tight* if $A = B$. The frame is *exact* if it ceases to be a frame whenever any single element is deleted from the sequence. ◊

If $\{x_n\}$ is a frame then $\sum |\langle x, x_n\rangle|^2$ is an absolutely convergent series of nonnegative real numbers. It therefore converges unconditionally by Lemma 2.4. Hence $\sum |\langle x, x_{\sigma(n)}\rangle|^2 = \sum |\langle x, x_n\rangle|^2 < \infty$ for any permutation σ of $\mathbf{N}$. As a consequence, every rearrangement of a frame is also a frame, and therefore we could use any countable set to index a frame if we wished.

Example 12.2. By the Plancherel formula (Theorem 1.20), every orthonormal basis $\{e_n\}$ is a tight frame with $A = B = 1$. Moreover, $\{e_n\}$ is an exact frame since if we delete any element e_m, then $\sum_{n \ne m} |\langle e_m, e_n\rangle|^2 = 0$, and therefore $\{e_n\}_{n \ne m}$ cannot be a frame. ◊

We will see in Theorem 12.21 that the class of exact frames for H coincides with the class of Riesz bases for H. Further, we shall see in Proposition 12.10 that even though an inexact frame is not a basis, the pseudo-Plancherel formula (12.1) implies that every element $x \in H$ can be expressed as $x = \sum c_n x_n$ with specified c_n. However, in this case the scalars c_n will not be unique.

The following example shows that tightness and exactness are distinct concepts.

Example 12.3. Let $\{e_n\}$ be an orthonormal basis for a separable Hilbert space H.

(a) $\{e_n\}$ is a tight exact frame for H with frame bounds $A = B = 1$.

(b) $\{e_1, e_1, e_2, e_2, e_3, e_3, \dots\}$ is a tight inexact frame with bounds $A = B = 2$, but it is not orthogonal and it is not a basis, although it does contain an orthonormal basis. Similarly, if $\{f_n\}$ is another orthonormal basis for H then $\{e_n\} \cup \{f_n\}$ is a tight inexact frame.

(c) $\{e_1, e_2/2, e_3/3, \dots\}$ is a complete orthogonal sequence and it is a basis for H, but it does not possess a lower frame bound and hence is not a frame.

(d) $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \dots\}$ is a tight inexact frame with bounds $A = B = 1$, and no nonredundant subsequence is a frame.

(e) $\{2e_1, e_2, e_3, \dots\}$ is a nontight exact frame with bounds $A = 1$, $B = 2$. $\diamondsuit$

We show now that all frames must be complete, although part (c) of the preceding example shows that there exist complete sequences which are not frames.

Lemma 12.4. *If $\{x_n\}$ is a frame for a Hilbert space H, then $\{x_n\}$ is complete in H.*

Proof. If $x \in H$ satisfies $\langle x, x_n\rangle = 0$ for all n, then $A\,\|x\|^2 \le \sum |\langle x, x_n\rangle|^2 = 0$. $\square$

As a consequence of this result, if H possesses a frame $\{x_n\}$ then it must be separable, since the set of all finite linear combinations $\sum_{n=1}^N c_n x_n$ with rational c_n (or rational real and imaginary parts if the c_n are complex) will form a countable, dense subset of H. Conversely, every separable Hilbert space does possess a frame since it possesses an orthonormal basis, which is a tight exact frames.

Example 12.5. Let $a, b > 0$ be fixed. If the collection $\{e^{2\pi i m b x} g(x - na)\}_{m,n\in\mathbf{Z}}$ of time-frequency translates of a single $g \in L^2(\mathbf{R})$ forms a frame for $L^2(\mathbf{R})$ then it is called a *Gabor frame*. Similarly, if the collection $\{a^{n/2} g(a^n x - mb)\}_{m,n\in\mathbf{Z}}$ of time-scale translates of $g \in L^2(\mathbf{R})$ forms a frame then it is called a *wavelet frame*. We refer to [Dau92], [HW89] for expository treatments of these types of frames. $\diamondsuit$

Recall from Definition 11.7 that $\{x_n\}$ is a *Bessel sequence* if $\sum |\langle x, x_n\rangle|^2 < \infty$ for every $x \in H$. By Lemma 11.8, or directly from the Uniform Boundedness Principle, every Bessel sequence must possess an upper frame bound $B > 0$, i.e.,

$$\forall\, x \in H, \quad \sum |\langle x, x_n\rangle|^2 \le B\|x\|^2.$$

The number B is sometimes called the *Bessel bound*, or simply the *upper frame bound* for $\{x_n\}$ (even though an arbitrary Bessel sequence need not satisfy a lower frame bound and therefore need not be a frame). In applications, a sequence which is a frame is often easily shown to be a Bessel sequence, while the lower frame bound is often more difficult to establish.

We now prove some basic properties of Bessel sequences and frames. Part (a) of the following lemma is proved in [DS52].

Lemma 12.6. *Let $\{x_n\}$ be a Bessel sequence with Bessel bound B.*

(a) *If $(c_n) \in \ell^2$ then $\sum c_n x_n$ converges unconditionally in H, and*

$$\Bigl\|\sum_n c_n x_n\Bigr\|^2 \;\le\; B \sum_n |c_n|^2.$$

(b) *$Ux = (\langle x, x_n\rangle)$ is a continuous mapping of H into ℓ^2, with $\|U\| \le B^{1/2}$. Its adjoint is the continuous mapping $U^*\colon \ell^2 \to H$ given by $U^*(c_n) = \sum c_n x_n$.*

(c) *If $\{x_n\}$ is a frame then U is injective and U^* is surjective.*

Proof. (a) Let F be any finite subset of $\mathbf{N}$. Then,

$$\begin{aligned}
\Bigl\|\sum_{n\in F} c_n x_n\Bigr\|^2 &= \sup_{\|y\|=1} \Bigl|\Bigl\langle \sum_{n\in F} c_n x_n,\ y\Bigr\rangle\Bigr|^2 && \text{by Theorem 1.16(b)}\\
&= \sup_{\|y\|=1} \Bigl|\sum_{n\in F} c_n \langle x_n, y\rangle\Bigr|^2 \\
&\le \sup_{\|y\|=1} \Bigl(\sum_{n\in F} |c_n|^2\Bigr)\Bigl(\sum_{n\in F} |\langle x_n, y\rangle|^2\Bigr) && \text{by Cauchy–Schwarz}\\
&\le \sup_{\|y\|=1} \Bigl(\sum_{n\in F} |c_n|^2\Bigr) B\|y\|^2 && \text{by definition of frame}\\
&= B \sum_{n\in F} |c_n|^2. && (12.2)
\end{aligned}$$

Since $\sum |c_n|^2$ is an absolutely and unconditionally convergent series of real numbers, it therefore follows from (12.2) and Theorem 2.8 that $\sum c_n x_n$ converges unconditionally in H. Setting $F = \{1, \dots, N\}$ in (12.2) and taking the limit as $N \to \infty$ then yields the desired inequality $\|\sum c_n x_n\|^2 \le B \sum |c_n|^2$.

(b) By definition of Bessel bound, we have $\|Ux\|_{\ell^2}^2 = \sum |\langle x, x_n\rangle|^2 \le B\,\|x\|^2$. Hence U is a continuous mapping of H into ℓ^2, and $\|U\| \le B^{1/2}$.

The adjoint $U^*\colon \ell^2 \to H$ of H is therefore well-defined and continuous, so we need only verify that it has the correct form. Now, if $(c_n) \in \ell^2$ then we know by part (a) that $\sum c_n x_n$ converges to an element of H. Therefore, given $x \in H$ we can compute

$$\langle x, U^*(c_n)\rangle \;=\; \langle Ux, (c_n)\rangle_{\ell^2} \;=\; \bigl\langle (\langle x, x_n\rangle),\ (c_n)\bigr\rangle_{\ell^2} \;=\; \sum_n \langle x, x_n\rangle\, \bar{c}_n \;=\; \Bigl\langle x,\ \sum_n c_n x_n\Bigr\rangle.$$

Hence $U^*(c_n) = \sum c_n x_n$.

(b) The fact that U is injective follows immediately from the fact that frames are complete. The fact that U^* is surjective follows from the fact that U is injective. $\square$

Definition 12.7. Let $\{x_n\}$ be a frame for a Hilbert space H.

(a) The *coefficient mapping* for $\{x_n\}$ is the continuous mapping $U: H \to \ell^2$ defined by $Ux = (\langle x, x_n\rangle)$ for $x \in H$.

(b) The *synthesis mapping* for $\{x_n\}$ is the continuous mapping $U^*: \ell^2 \to H$ defined by $U^*(c_n) = \sum c_n x_n$ for $(c_n) \in \ell^2$.

(c) The *frame operator* for $\{x_n\}$ is the continuous mapping $S: H \to H$ defined by

$$Sx \;=\; U^*Ux \;=\; \sum_n \langle x, x_n\rangle\, x_n, \qquad x \in H. \quad \diamondsuit$$

Proposition 12.8. [DS52]. *Given a sequence $\{x_n\}$ in a Hilbert space H, the following statements are equivalent.*

(a) *$\{x_n\}$ is a frame with frame bounds A, B.*

(b) *$Sx = \sum \langle x, x_n\rangle\, x_n$ is a positive, bounded, linear mapping of H into H which satisfies $AI \le S \le BI$.*

Proof. (a) $\Rightarrow$ (b). Assume that $\{x_n\}$ is a frame. Then $S = U^*U$ is continuous by Lemma 12.6. In fact, we have $\|S\| \le \|U^*\|\,\|U\| \le B$. Note that

$$\langle AIx, x\rangle \;=\; A\,\|x\|^2, \qquad \langle Sx, x\rangle \;=\; \sum_n |\langle x, x_n\rangle|^2, \qquad \langle BIx, x\rangle \;=\; B\,\|x\|^2. \tag{12.3}$$

Therefore, $\langle AIx, x\rangle \le \langle Sx, x\rangle \le \langle BIx, x\rangle$ by definition of frame, so $AI \le S \le BI$. Additionally, $\langle Sx, x\rangle \ge 0$ for every x, so S is a positive operator.

(b) $\Rightarrow$ (a). Assume that statement (b) holds. Then $\langle AIx, x\rangle \le \langle Sx, x\rangle \le \langle BIx, x\rangle$ for every $x \in H$. By (12.3), this implies that $\{x_n\}$ is a frame for H. $\square$

Our next goal is to show that the frame operator S is a topological isomorphism of H onto itself. We will require the following lemma. To motivate this lemma, note that if $T: H \to H$ is a positive definite operator, i.e., $\langle Tx, x\rangle > 0$ for all $x \ne 0$, then $(x, y) = \langle Tx, y\rangle$ defines an inner product on H that is equivalent to the original inner product. If we let $|||x||| = (x, x)^{1/2}$ denote the corresponding induced norm, then the Cauchy–Schwarz inequality applied to $(\cdot, \cdot)$ states that

$$|\langle Tx, y\rangle|^2 \;=\; |(x, y)|^2 \;\le\; |||x|||^2\; |||y|||^2 \;=\; (x, x)\,(y, y) \;=\; \langle Tx, x\rangle\,\langle Ty, y\rangle.$$

The following lemma states that this inequality remains valid even if T is only assumed to be a positive operator, rather than positive definite. In this case, $(x, x) = \langle Tx, x\rangle \ge 0$ for all x, but we may have $(x, x) = 0$ when $x \ne 0$. Hence $(\cdot, \cdot)$ need not be an inner product in this case. However, the proof of the Cauchy–Schwarz inequality does adapt to this more general situation.

Lemma 12.9 (Generalized Cauchy–Schwarz). *Let H be a Hilbert space. If $T\colon H \to H$ is a positive operator, then*

$$\forall\, x, y \in H, \quad |\langle Tx, y\rangle|^2 \;\le\; \langle Tx, x\rangle\, \langle Ty, y\rangle. \quad \diamondsuit$$

Proposition 12.10. [DS52]. *If $\{x_n\}$ is a frame for a Hilbert space H, then the following statements hold.*

(a) *The frame operator S is a topological isomorphism of H onto itself. Moreover, S^{-1} satisfies $B^{-1}I \le S^{-1} \le A^{-1}I$.*

(b) *$\{S^{-1}x_n\}$ is a frame for H, with frame bounds B^{-1}, A^{-1}.*

(c) *The following series converge unconditionally for each $x \in H$:*

$$x \;=\; \sum_n \langle x, S^{-1}x_n\rangle\, x_n \;=\; \sum_n \langle x, x_n\rangle\, S^{-1}x_n. \tag{12.4}$$

(d) *If the frame is tight, i.e., $A = B$, then $S = AI$, $S^{-1} = A^{-1}I$, and*

$$\forall\, x \in H, \quad x \;=\; A^{-1} \sum \langle x, x_n\rangle x_n.$$

Proof. (a) We know that S is continuous since $S = U^*U$ and U is continuous. Further, it follows from $AI \le S \le BI$ that $A\,\|x\|^2 = \langle AIx, x\rangle \le \langle Sx, x\rangle \le \|Sx\|\,\|x\|$. Hence,

$$\forall\, x \in H, \quad A\,\|x\| \;\le\; \|Sx\|. \tag{12.5}$$

This implies immediately that S is injective, and that $S^{-1}\colon \mathrm{Range}(S) \to H$ is continuous. Hence, if we show that S is surjective then it follows that S is a topological isomorphism.

Before showing that $\mathrm{Range}(S) = H$, we will show that $\mathrm{Range}(S)$ is closed. Suppose that $y_n \in \mathrm{Range}(S)$ and that $y_n \to y \in H$. Then $y_n = Sx_n$ for some $x_n \in H$. Hence $\{Sx_n\}$ is a Cauchy sequence in H. However, by (12.5), we have $A\,\|x_m - x_n\| \le \|Sx_m - Sx_n\|$, so $\{x_n\}$ is Cauchy as well. Therefore $x_n \to x$ for some $x \in H$. Since S is continuous, we therefore have $y_n = Sx_n \to Sx$. Since we also have $y_n \to y$, we conclude that $y = Sx \in \mathrm{Range}(S)$, so $\mathrm{Range}(S)$ is closed.

Now we will show that $\mathrm{Range}(S) = H$. Suppose that $y \in H$ was orthogonal to $\mathrm{Range}(S)$, i.e., $\langle y, Sx\rangle = 0$ for every $x \in H$. Then $A\,\|y\|^2 = \langle AIy, y\rangle \le \langle Sy, y\rangle = 0$, so $y = 0$. Since $\mathrm{Range}(S)$ is a closed subspace of H, it follows that $\mathrm{Range}(S) = H$. Thus S is surjective, and therefore is a topological isomorphism.

Finally, we will show that S^{-1} satisfies $B^{-1}I \le S^{-1} \le A^{-1}I$. First, note that S^{-1} is positive since S is positive. This also follows from the computation

$$0 \;\le\; A\,\|S^{-1}x\|^2 \;=\; \langle AI(S^{-1}x), S^{-1}x\rangle \;\le\; \langle S(S^{-1}x), S^{-1}x\rangle \;=\; \langle x, S^{-1}x\rangle \;\le\; \|x\|\,\|S^{-1}x\|.$$

As a consequence, $\|S^{-1}\| \le A^{-1}$. Hence $\langle S^{-1}x, x\rangle \le \|S^{-1}x\|\,\|x\| \le A^{-1}\,\|x\|^2 = \langle A^{-1}Ix, x\rangle$, so $S^{-1} \le A^{-1}I$. Lastly, by Lemma 12.9,

$$\begin{aligned}\|x\|^4 \;=\; \langle x, x\rangle^2 \;=\; \langle S^{-1}(Sx), x\rangle^2 \;&\le\; \langle S^{-1}(Sx), Sx\rangle\, \langle S^{-1}x, x\rangle \\ &=\; \langle x, Sx\rangle\, \langle S^{-1}x, x\rangle \\ &\le\; B\,\|x\|^2\, \langle S^{-1}x, x\rangle.\end{aligned}$$

Therefore $\langle S^{-1}x, x\rangle \ge B^{-1}\,\|x\|^2 = \langle B^{-1}Ix, x\rangle$, so $S^{-1} \ge B^{-1}I$.

(b) The operator S^{-1} is self-adjoint since it is positive. Therefore,

$$\begin{aligned}\sum_n \langle x, S^{-1}x_n\rangle\, S^{-1}x_n &= \sum_n \langle S^{-1}x, x_n\rangle\, S^{-1}x_n \\ &= S^{-1}\Bigl(\sum_n \langle S^{-1}x, x_n\rangle\, x_n\Bigr) = S^{-1}S(S^{-1}x) = S^{-1}x.\end{aligned}$$

Since we also have $B^{-1}I \le S^{-1} \le A^{-1}I$, it therefore follows from Proposition 12.8 that $\{S^{-1}x_n\}$ is a frame.

(c) We compute

$$x = S(S^{-1}x) = \sum_n \langle S^{-1}x, x_n\rangle\, x_n = \sum_n \langle x, S^{-1}x_n\rangle\, x_n$$

and

$$x = S^{-1}(Sx) = S^{-1}\Bigl(\sum_n \langle x, x_n\rangle\, x_n\Bigr) = \sum_n \langle x, x_n\rangle\, S^{-1}x_n.$$

The unconditionality of the convergence follows from the fact that $\{x_n\}$ and $\{S^{-1}x_n\}$ are both frames.

(d) Follows immediately from parts (a)–(c). ☐

Definition 12.11. Let $\{x_n\}$ be a frame with frame operator S. Then $\{S^{-1}x_n\}$ is the *dual frame* of $\{x_n\}$.

We now prove some results relating to the uniqueness of the series expressions in (12.4). The following proposition shows that among all choices of scalars (c_n) for which $x = \sum c_n x_n$, the scalars $c_n = \langle x, S^{-1}x_n\rangle$ have the minimal ℓ^2-norm.

Proposition 12.12. [DS52]. *Let $\{x_n\}$ be a frame for a Hilbert space H, and let $x \in H$. If $x = \sum c_n x_n$ for some scalars (c_n), then*

$$\sum_n |c_n|^2 = \sum_n |\langle x, S^{-1}x_n\rangle|^2 + \sum_n |\langle x, S^{-1}x_n\rangle - c_n|^2.$$

In particular, the sequence $(\langle x, S^{-1}x_n\rangle)$ has the minimal ℓ^2-norm among all such sequences (c_n).

Proof. By (12.4), we have $x = \sum a_n x_n$, where $a_n = \langle x, S^{-1}x_n\rangle$. Let (c_n) be any sequence of scalars such that $x = \sum c_n x_n$. Since $\sum |a_n|^2 < \infty$, we may assume without loss of generality that $\sum |c_n|^2 < \infty$. Then $(c_n) \in \ell^2$, and we have

$$\langle x, S^{-1}x\rangle = \Bigl\langle \sum_n a_n x_n,\ S^{-1}x\Bigr\rangle = \sum_n a_n \langle S^{-1}x_n, x\rangle = \sum_n a_n \bar{a}_n = \bigl\langle (a_n),\ (a_n)\bigr\rangle_{\ell^2}$$

and

$$\langle x, S^{-1}x\rangle = \Bigl\langle \sum_n c_n x_n,\ S^{-1}x\Bigr\rangle = \sum_n c_n \langle S^{-1}x_n, x\rangle = \sum_n c_n \bar{a}_n = \bigl\langle (c_n),\ (a_n)\bigr\rangle_{\ell^2}.$$

Therefore $(c_n - a_n)$ is orthogonal to (a_n) in ℓ^2, whence

$$\|(c_n)\|_{\ell^2}^2 \;=\; \|(c_n - a_n) + (a_n)\|_{\ell^2}^2 \;=\; \|(c_n - a_n)\|_{\ell^2}^2 \;+\; \|(a_n)\|_{\ell^2}^2. \quad \square$$

The following result will be play an important role in characterizing the class of exact frames.

Proposition 12.13. [DS52]. *Let $\{x_n\}$ be a frame for a Hilbert space H.*

(a) *For each m,*

$$\sum_{n \neq m} |\langle x_m, S^{-1}x_n\rangle|^2 \;=\; \frac{1 \;-\; |\langle x_m, S^{-1}x_m\rangle|^2 \;-\; |1 - \langle x_m, S^{-1}x_m\rangle|^2}{2}. \tag{12.6}$$

(b) *If $\langle x_m, S^{-1}x_m\rangle = 1$, then $\langle x_m, S^{-1}x_n\rangle = 0$ for $n \neq m$.*

(c) *The removal of a vector from a frame leaves either a frame or an incomplete set. In fact,*

$$\langle x_m, S^{-1}x_m\rangle \neq 1 \;\implies\; \{x_n\}_{n\neq m} \text{ is a frame},$$

$$\langle x_m, S^{-1}x_m\rangle = 1 \;\implies\; \{x_n\}_{n\neq m} \text{ is incomplete}.$$

Proof. (a) Fix any m, and let $a_n = \langle x_m, S^{-1}x_n\rangle$. Then $x_m = \sum a_n x_n$ by (12.4). However, we also have $x_m = \sum \delta_{mn} x_n$, so Proposition 12.12 implies that

$$\begin{aligned} 1 \;=\; \sum_n |\delta_{mn}|^2 \;&=\; \sum_n |a_n|^2 \;+\; \sum_n |a_n - \delta_{mn}|^2 \\ &=\; |a_m|^2 \;+\; \sum_{n\neq m} |a_n|^2 \;+\; |a_m - 1|^2 \;+\; \sum_{n \neq m} |a_n|^2. \end{aligned}$$

Therefore,

$$\sum_{n\neq m} |a_n|^2 \;=\; \frac{1 - |a_m|^2 - |a_m - 1|^2}{2}.$$

(b) Suppose that $\langle x_m, S^{-1}x_m\rangle = 1$. Then we have $\sum_{n\neq m} |\langle x_m, S^{-1}x_n\rangle|^2 = 0$ by (12.6). Hence $\langle S^{-1}x_m, x_n\rangle = 0$ for $n \neq m$.

(c) Suppose that $\langle x_m, S^{-1}x_m\rangle = 1$. Then by part (b), $S^{-1}x_m$ is orthogonal to x_n for every $n \neq m$. However, $S^{-1}x_m \neq 0$ since $\langle S^{-1}x_m, x_m\rangle = 1 \neq 0$. Therefore $\{x_n\}_{n\neq m}$ is incomplete in this case.

On the other hand, suppose that $\langle x_m, S^{-1}x_m\rangle \neq 1$, and set $a_n = \langle x_m, S^{-1}x_n\rangle$. We have $x_m = \sum a_n x_n$ by (12.4). Since $a_m \neq 1$, we therefore have $x_m = \frac{1}{1-a_m} \sum_{n\neq m} a_n x_n$. Hence, for each $x \in H$,

$$|\langle x, x_m\rangle|^2 \;=\; \left|\frac{1}{1 - a_m} \sum_{n\neq m} a_n \,\langle x, x_n\rangle\right|^2 \;\leq\; C \sum_{n\neq m} |\langle x, x_n\rangle|^2,$$

where $C = |1 - a_m|^{-2} \sum_{n \neq m} |a_n|^2$. Therefore,

$$\sum_n |\langle x, x_n \rangle|^2 \;=\; |\langle x, x_m \rangle|^2 \;+\; \sum_{n \neq m} |\langle x, x_n \rangle|^2 \;\leq\; (1 + C) \sum_{n \neq m} |\langle x, x_n \rangle|^2.$$

Hence,

$$\frac{A}{1 + C} \, \|x\|^2 \;\leq\; \frac{1}{1 + C} \sum_n |\langle x, x_n \rangle|^2 \;\leq\; \sum_{n \neq m} |\langle x, x_n \rangle|^2 \;\leq\; B \, \|x\|^2.$$

Thus $\{x_n\}_{n \neq m}$ is a frame with bounds $A/(1 + C)$, B. □

As a consequence, we find that a frame is exact if and only if it is biorthogonal to its dual frame.

Corollary 12.14. *If $\{x_n\}$ is a frame for a Hilbert space H, then the following statements are equivalent.*

(a) *$\{x_n\}$ is an exact frame.*

(b) *$\{x_n\}$ and $\{S^{-1} x_n\}$ are biorthogonal.*

(c) *$\langle x_n, S^{-1} x_n \rangle = 1$ for all n.*

As a consequence, if the frame is tight, i.e., $A = B$, then the following statements are equivalent.

(a') *$\{x_n\}$ is an exact frame.*

(b') *$\{x_n\}$ is an orthogonal sequence.*

(c') *$\|x_n\|^2 = A$ for all n.*

Proof. (a) ⇒ (c). If $\{x_n\}$ is an exact frame, then, by definition, $\{x_n\}_{n \neq m}$ is not a frame for any m. It therefore follows from Proposition 12.13 that $\langle x_m, S^{-1} x_m \rangle = 1$ for every m.

(c) ⇒ (a). Suppose that $\langle x_m, S^{-1} x_m \rangle = 1$ for every m. Proposition 12.13 then implies that $\{x_n\}_{n \neq m}$ is not complete, and hence is not a frame. Therefore $\{x_n\}$ is exact by definition.

(b) ⇒ (c). This follows immediately from the definition of biorthogonality.

(c) ⇒ (b). This follows immediately from Proposition 12.13(b). □

In Example 12.3(d), we constructed a frame that is not norm-bounded below. The following result shows that all frames are norm-bounded above, and that only inexact frames can be unbounded below.

Proposition 12.15. *Let $\{x_n\}$ be a frame for a Hilbert space H. Then the following statements hold.*

(a) *$\{x_n\}$ is norm-bounded above, and $\sup \|x_n\|^2 \leq B$.*

(b) *If $\{x_n\}$ is exact then it is norm-bounded below, and $A \leq \inf \|x_n\|^2$.*

Proof. (a) With m fixed, we have

$$\|x_m\|^4 \;=\; |\langle x_m, x_m \rangle|^2 \;\leq\; \sum_n |\langle x_m, x_n \rangle|^2 \;\leq\; B \, \|x_m\|^2.$$

(b) If $\{x_n\}$ is an exact, then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal by Corollary 12.15. Therefore, for each fixed m,

$$A\,\|S^{-1}x_m\|^2 \;\le\; \sum_n |\langle S^{-1}x_m, x_n\rangle|^2 \;=\; |\langle S^{-1}x_m, x_m\rangle|^2 \;\le\; \|S^{-1}x_m\|^2\,\|x_m\|^2.$$

Since $\{x_n\}$ is exact we must have $x_m \neq 0$. Since S is a topological isomorphism, we therefore have $S^{-1}x_m \neq 0$ as well, so we can divide by $\|S^{-1}x_m\|^2$ to obtain the desired inequality. □

We collect now some remarks on the convergence of $\sum c_n x_n$ for arbitrary sequences of scalars.

Recall that if $\{x_n\}$ is a frame and $\sum |c_n|^2 < \infty$, then $\sum c_n x_n$ converges (Lemma 12.6). The following example shows that the converse is not true in general.

Example 12.16. [Hei90]. *$\sum c_n x_n$ converges* $\not\Rightarrow$ $\sum |c_n|^2 < \infty$.

Let $\{x_n\}$ be any frame which includes infinitely many zero elements. Let $c_n = 1$ whenever $x_n = 0$, and let $c_n = 0$ when $x_n \neq 0$. Then $\sum c_n x_n = 0$, even though $\sum |c_n|^2 = \infty$.

Less trivially, let $\{e_n\}$ be an orthonormal basis for a Hilbert space H. Define $f_n = n^{-1}e_n$ and $g_n = (1-n^{-2})^{1/2}e_n$. Then $\{f_n\} \cup \{g_n\}$ is a tight frame with $A = B = 1$. Let $x = \sum n^{-1}e_n$. This is an element of H since $\sum n^{-2} < \infty$. However, in terms of the frame $\{f_n\} \cup \{g_n\}$ we have $x = \sum (1 \cdot f_n + 0 \cdot g_n)$, although $\sum (1^2 + 0^2) = \infty$. ♢

By Lemma 12.6, if $(c_n) \in \ell^2$ then $\sum c_n x_n$ converges unconditionally. The preceding example shows that $\sum c_n x_n$ may converge even though $(c_n) \notin \ell^2$. However, we show next that if $\{x_n\}$ is norm-bounded below, then $\sum c_n x_n$ converges unconditionally exactly for $(c_n) \in \ell^2$.

Proposition 12.17. [Hei90]. *If $\{x_n\}$ be a frame that is norm-bounded below, then*

$$\sum_n |c_n|^2 < \infty \quad\Longleftrightarrow\quad \sum_n c_n x_n \text{ converges unconditionally.}$$

Proof. $\Rightarrow$. This is the content of Lemma 12.6.

$\Leftarrow$. Assume that $\sum c_n x_n$ converges unconditionally. Then Orlicz's Theorem (Theorem 3.1) implies that $\sum |c_n|^2\,\|x_n\|^2 = \sum \|c_n x_n\|^2 < \infty$. Since $\{x_n\}$ is norm-bounded below, it therefore follows that $\sum |c_n|^2 < \infty$. □

We shall see in Example 12.23 that, for an exact frame, $\sum c_n x_n$ converges if and only if it converges unconditionally. The next example shows that, for an inexact frame, $\sum c_n x_n$ may converge conditionally, even if the frame is norm-bounded below.

Example 12.18. [Hei90]. We will construct a frame $\{x_n\}$ which is norm-bounded below and a sequence of scalars (c_n) such that $\sum c_n x_n$ converges but $\sum |c_n|^2 = \infty$.

Let $\{e_n\}$ be any orthonormal basis for a separable Hilbert space H. Then $\{e_1, e_1, e_2, e_2, \dots\}$ is a frame that is norm-bounded below. The series

$$e_1 \;-\; e_1 \;+\; \frac{e_2}{\sqrt{2}} \;-\; \frac{e_2}{\sqrt{2}} \;+\; \frac{e_3}{\sqrt{3}} \;-\; \frac{e_3}{\sqrt{3}} \;+\; \cdots \tag{12.7}$$

converges strongly in H to 0. However, the series

$$e_1 + e_1 + \frac{e_2}{\sqrt{2}} + \frac{e_2}{\sqrt{2}} + \frac{e_3}{\sqrt{3}} + \frac{e_3}{\sqrt{3}} + \cdots$$

does not converge. Therefore, the series in (12.7) converges conditionally by Theorem 2.8. Since $(n^{-1/2}) \notin \ell^2$, the conditionality of the convergence also follows from Proposition 12.17. $\diamondsuit$

In the remainder of this chapter, we will determine the exact relationship between frames and bases.

Proposition 12.19. *An inexact frame is not a basis.*

Proof. Assume that $\{x_n\}$ is an inexact frame. Then, by definition, $\{x_n\}_{n\neq m}$ is a frame for some m, and is therefore complete. However, no proper subset of a basis can be complete, so $\{x_n\}$ cannot be a basis. Additionally, we have $x_m = \sum \langle x_m, S^{-1}x_n\rangle\, x_n$ by (12.4), and also $x_m = \sum \delta_{mn} x_n$. By Proposition 12.13, the fact that $\{x_n\}_{n\neq m}$ is a frame implies that $\langle x_m, S^{-1}x_n\rangle \neq 1$. Hence these are two distinct representations of x_m in terms of the frame elements, so $\{x_n\}$ cannot be a basis. $\square$

We show now that frames are preserved by topological isomorphisms (compare Lemmas 4.12, 9.3, and 11.3 for bases, unconditional bases, or Riesz bases, respectively).

Lemma 12.20. *Frames are preserved by topological isomorphisms. That is, if $\{x_n\}$ is a frame for a Hilbert space H and $T\colon H \to K$ is a topological isomorphism, then $\{Tx_n\}$ is a frame for K. In this case, we have the following additional statements:*

(a) *If $\{x_n\}$ has frame bounds A, B, then $\{Tx_n\}$ has frame bounds $A\,\|T^{-1}\|^{-2}$, $B\,\|T\|^2$.*

(b) *If $\{x_n\}$ has frame operator S, then $\{Tx_n\}$ has frame operator TST^*.*

(c) *$\{x_n\}$ is exact if and only if $\{Tx_n\}$ is exact.*

Proof. Note that for each $y \in K$,

$$TST^*y \;=\; T\biggl(\sum_n \langle T^*y, x_n\rangle\, x_n\biggr) \;=\; \sum_n \langle y, Tx_n\rangle\, Tx_n.$$

Therefore, the fact that $\{Tx_n\}$ is a frame and both statements (a) and (b) will follow from Proposition 12.8 if we show that $A\,\|T^{-1}\|^{-2} I \le TST^* \le B\,\|T\|^2 I$. Now, if $y \in H$ then $\langle TST^*y, y\rangle = \langle S(T^*y), (T^*y)\rangle$, so it follows from $AI \le S \le BI$ that

$$A\,\|T^*y\|^2 \;\le\; \langle TST^*y, y\rangle \;\le\; B\,\|T^*y\|^2. \tag{12.8}$$

Further, since T is a topological isomorphism, we have

$$\frac{\|y\|}{\|T^{-1}\|} \;=\; \frac{\|y\|}{\|{T^*}^{-1}\|} \;\le\; \|T^*y\| \;\le\; \|T^*\|\,\|y\| \;=\; \|T\|\,\|y\|. \tag{12.9}$$

Combining (12.8) and (12.9), we find that

$$\frac{A\,\|y\|^2}{\|T^{-1}\|^2} \;\le\; \langle TST^*y, y\rangle \;\le\; B\,\|T\|^2\,\|y\|^2,$$

which is equivalent to the desired statement $A\,\|T^{-1}\|^{-2}I \le TST^* \le B\,\|T\|^2 I$.

Finally, statement (c) regarding exactness follows from the fact that topological isomorphisms preserve complete and incomplete sequences. □

We can now show that the class of exact frames for H coincides with the class of bounded unconditional bases for H. By Theorem 11.9, this further coincides with the class of Riesz bases for H. The statement and a different proof of the following result can be found in [You80].

Theorem 12.21. *Let $\{x_n\}$ be a sequence in a Hilbert space H. Then $\{x_n\}$ is an exact frame for H if and only if it is a bounded unconditional basis for H.*

Proof. $\Rightarrow$. Assume that $\{x_n\}$ is an exact frame for H. Then $\{x_n\}$ is norm-bounded both above and below by Proposition 12.15. We have from (12.4) that $x = \sum \langle x, S^{-1}x_n\rangle\, x_n$ for all x, with unconditional convergence of this series. To see that this representation is unique, suppose that we also had $x = \sum c_n x_n$. By Corollary 12.5, $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal sequences. Therefore,

$$\langle x, S^{-1}x_m\rangle \;=\; \Big\langle \sum_n c_n x_n,\; S^{-1}x_m \Big\rangle \;=\; \sum_n c_n\, \langle x_n, S^{-1}x_m\rangle \;=\; \sum_n c_n \delta_{nm} \;=\; c_m.$$

Hence the representation $x = \sum \langle x, S^{-1}x_n\rangle\, x_n$ is unique, so $\{x_n\}$ is a bounded unconditional basis for H.

$\Leftarrow$. Assume that $\{x_n\}$ is a bounded unconditional basis for H. Then $\{x_n\}$ is a Riesz basis by Theorem 11.9. Therefore, by definition of Riesz basis, there exists an orthonormal basis $\{e_n\}$ for H and a topological isomorphism $T\colon H \to H$ such that $Te_n = x_n$ for all n. However, $\{e_n\}$ is an exact frame and exact frames are preserved by topological isomorphisms (Lemma 12.20), so $\{x_n\}$ must be an exact frame for H. □

We can exhibit directly the topological isomorphism T used in the proof of Theorem 12.21. Since S is a positive operator that is a topological isomorphism of H onto itself, it has a square root $S^{1/2}$ that is a positive topological isomorphism of H onto itself [Wei80, Theorem 7.20]. Similarly, S^{-1} has a square root $S^{-1/2}$, and it is easy to verify that $(S^{1/2})^{-1} = S^{-1/2}$. Since $\{x_n\}$ is exact, $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal by Corollary 12.15. Therefore,

$$\langle S^{-1/2}x_m, S^{-1/2}x_n\rangle \;=\; \langle x_m, S^{-1/2}S^{-1/2}x_n\rangle \;=\; \langle x_m, S^{-1}x_n\rangle \;=\; \delta_{mn}.$$

Thus $\{S^{-1/2}x_n\}$ is an orthonormal sequence. Moreover, it is complete since topological isomorphisms preserve complete sequences. Therefore, $\{S^{-1/2}x_n\}$ is an orthonormal basis for H by Theorem 1.20, and the topological isomorphism $T = S^{1/2}$ maps this orthonormal basis onto the frame $\{x_n\}$.

We can consider the sequence $\{S^{-1/2}x_n\}$ for any frame, not just exact frames. If $\{x_n\}$ is inexact then $\{S^{-1/2}x_n\}$ will not be an orthonormal basis for H, but we show next that it will be a tight frame for H.

Corollary 12.22. *Every frame is equivalent to a tight frame. That is, if $\{x_n\}$ is a frame with frame operator S then $S^{-1/2}$ is a positive topological isomorphism of H onto itself, and $\{S^{-1/2}x_n\}$ is a tight frame with bounds $A = B = 1$.*

Proof. Since $S^{-1/2}$ is a topological isomorphism, it follows from Lemma 12.20 that $\{S^{-1/2}x_n\}$ is a frame for H. Note that for each $x \in H$ we have

$$\sum_n \langle x, S^{-1/2}x_n \rangle \, S^{-1/2}x_n \;=\; S^{-1/2}SS^{-1/2}x \;=\; x \;=\; Ix.$$

Proposition 12.8 therefore implies that the frame is tight and has frame bounds $A = B = 1$. □

Example 12.23. If $\{x_n\}$ is an exact frame then it is a Riesz basis for H. Hence by Theorem 11.9 and Proposition 12.17, we have that

$$\sum_n |c_n|^2 < \infty \quad\Longleftrightarrow\quad \sum_n c_n x_n \text{ converges} \quad\Longleftrightarrow\quad \sum_n c_n x_n \text{ converges unconditionally.}$$

By Example 12.18, these equivalences may fail if the frame is inexact. However, we can construct an inexact frame for which these equivalences remain valid.

Let $\{e_n\}$ be any orthonormal basis for a separable Hilbert space H, and consider the frame $\{x_n\} = \{e_1, e_1, e_2, e_3, \dots\}$. Then the series $\sum c_n x_n$ converges if and only if $\sum |c_n|^2 < \infty$ since $\{x_n\}$ is obtained from an orthonormal basis by the addition of a single element. Further, since $\{x_n\}$ is norm-bounded below, it follows from Proposition 12.17 that $\sum |c_n|^2 < \infty$ if and only if $\sum c_n x_n$ converges unconditionally. ◊

The frame $\{e_1, e_1, e_2, e_3, \dots\}$ considered in Example 12.23 consists of an orthonormal basis plus one additional element. Holub [Hol94] has characterized those frames $\{x_n\}$ which consist of a Riesz basis plus finitely many elements.

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