Seminorms for the Weak & Weak* Topologies

Example E.7 (The Weak Topology). Let X be any normed linear space. The norm induces one topology on X, but it is not the only natural topology. Each element μ of the dual space X^* provides us with a seminorm ρ_μ on X, defined by

$$\rho_\mu(x) \;=\; |\langle x, \mu\rangle|, \qquad x \in X.$$

The topology induced by the family of seminorms $\{\rho_\mu\}_{\mu\in X^*}$ is called the *weak topology* on X. Since there are uncountably many seminorms (and no obvious way to reduce to a countable collection in general), in order to relate the topology to a convergence criterion we must use nets instead of sequences. Writing $x_i \xrightarrow{\mathrm{w}} x$ to denote convergence of a net $\{x_i\}_{i\in I}$ in X with respect to the weak topology, the convergence criterion is

$$x_i \xrightarrow{\mathrm{w}} x \qquad \Longleftrightarrow \qquad \rho_\mu(x - x_i) \to 0 \;\text{ for all } \mu \in X^*.$$

Equivalently,

$$x_i \xrightarrow{\mathrm{w}} x \qquad \Longleftrightarrow \qquad \langle x_i, \mu\rangle \to \langle x, \mu\rangle \;\text{ for all } \mu \in X^*.$$

Norm convergence implies weak convergence, but the converse fails in general. In terms of topologies, every set that is open with respect to the weak topology is also open with respect to the norm topology, but not conversely. In essence, it is "easier" to converge in the weak topology because there are fewer open sets in that topology (hence the weak topology is *weaker* than the norm topology).

Note that if H is a Hilbert space, then, by the Riesz Representation Theorem, each continuous linear functional on H corresponds to the inner product with respect to an element of H. Hence, $x_i \xrightarrow{\mathrm{w}} x$ in H if $\langle x_i, y\rangle \to \langle x, y\rangle$ for each $y \in H$.

Example E.8 (The Weak Topology).* Let X be any normed linear space. Then its dual space X^* is also a normed linear space, and hence has a topology defined by its norm, as well as a weak topology as described above. However, there is also a third natural topology associated with X^*. Each element x in X determines a seminorm ρ_x on X^* by

$$\rho_x(\mu) \;=\; |\langle x, \mu\rangle|, \qquad \mu \in X^*.$$

The topology this family of seminorms $\{\rho_x\}_{x\in X}$ induces is called the *weak* topology* on X, and convergence with respect to this topology is denoted by $\mu_i \xrightarrow{\mathrm{w}^*} \mu$. Explicitly, if $\{\mu_i\}_{i\in I}$ is a net in X^* then the convergence criterion corresponding to the weak* topology is

$$\mu_i \xrightarrow{\mathrm{w}^*} \mu \qquad \Longleftrightarrow \qquad \rho_x(\mu - \mu_i) \to 0 \;\text{ for all } x \in X.$$

Equivalently,

$$\mu_i \xrightarrow{w^*} \mu \quad \iff \quad \langle x, \mu_i \rangle \to \langle x, \mu \rangle \text{ for all } x \in X.$$

Since $X \subseteq X^{**}$, the family of seminorms associated with the weak* topology includes only some of the seminorms associated with the weak topology. Hence weak convergence in X^* implies weak* convergence in X^*. Of course, if X is reflexive, then $X = X^{**}$ and the weak and weak* topologies on X^* are the same.

Relation between norm & weak topologies

It is easy to see that norm convergence implies weak convergence. This implies a relation between the norm & weak topologies. Let $\mathcal{T}^n$ be the norm topology & $\mathcal{T}^\omega$ the weak topology. Then

$$\left(\begin{matrix}\text{norm convergence implies} \\ \text{weak convergence}\end{matrix}\right) \iff \mathcal{T}^\omega \subseteq \mathcal{T}^n$$

Let us look just at the "$\Leftarrow$" implication. Suppose that $\mathcal{T}^\omega \subseteq \mathcal{T}^n$, i.e., every set that is open in the weak topology is open in the norm topology. Suppose that $x_n \to x$, i.e., convergence in norm. Then by definition,

$$\forall \text{ norm-open } U \text{ with } x \in U, \quad \exists N > 0 \text{ s.t.}$$

$$n > N \implies x_n \in U$$

(usually we just consider open balls of radius ε).

Then since every weakly open set is norm-open, we have exactly the same statement for weakly open sets:

$$\forall \text{ weakly-open } U \text{ with } x \in U, \quad \exists N > 0 \text{ s.t.}$$

$$n > N \implies x_n \in U.$$

Hence $x_n \xrightarrow{w} x$.

The same argument applies if we use nets instead of sequences.

Example: ℓ^1

As in any Banach space, norm convergence implies weak convergence, which is equivalent to saying that the weak topology is a subset (is "weaker than") of the norm topology.

However, it can be shown that ℓ^1 has the interesting property that

$$x_n \xrightarrow{w} x \text{ in } \ell^1 \quad \Rightarrow \quad x_n \to x \text{ in } \ell^1$$

The proof is more than an exercise, see Conway's functional analysis text. Hence norm convergence and weak convergence of sequences in ℓ^1 are equivalent! Doesn't this imply that the norm & weak topologies on ℓ^1 are equal? Surprise: No! It can be shown that these two topologies are <u>not</u> equal.

The subtlety is that if a topology is not induced from a metric, then we must be careful to define convergence in terms of nets rather than just sequences. Weak convergence of a net in l^1 does not imply norm convergence.

This is *usually* just a technicality, so we often restrict our attention to convergence of sequences. However, it is good to be aware of the distinction.