E.2 Topological Vector Spaces

General topologies were reviewed in Section A.6. Now we will consider vector spaces that have "nice" topologies, including those that are generated by families of seminorms.

E.2.1 Base for a Topology

A base for a topology is a set of "building blocks" for the topology, playing a role analogous to the one played by the collection of open balls in a metric space.

Definition E.9 (Base for a Topology). Let $\mathcal{T}$ be a topology on a set X. A *base* for the topology is a collection of open sets $\mathcal{B} \subseteq \mathcal{T}$ such that for any open set $U \in \mathcal{T}$ and any vector $x \in U$, there exists a base element $B \in \mathcal{B}$ that contains x and is contained in U, i.e.,

$$\forall\, U \in \mathcal{T}, \quad \forall\, x \in U, \quad \exists\, B \in \mathcal{B} \ \text{ such that } \ x \in B \subseteq U.$$

Consequently, if $\mathcal{B}$ is a base for a topology, then a set U is open if and only if

$$U \;=\; \bigcup_{x \in U} \bigl\{B_x : B_x \in \mathcal{B} \text{ and } x \in B_x \subseteq U\bigr\}.$$

In particular, if X is a metric space, the collection of open balls in X forms a base for the topology induced by the metric.

Remark E.10. If $\mathcal{E}$ is an arbitrary collection of subsets of X, then the smallest topology that contains $\mathcal{E}$ is called the *topology generated by* $\mathcal{E}$, denoted $\mathcal{T}(\mathcal{E})$, see Exercise A.39. Each element of $\mathcal{T}(\mathcal{E})$ can be written as a union of finite intersections of elements of $\mathcal{E}$. If contrast, if $\mathcal{B}$ is a base for a topology $\mathcal{T}$, then each element of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$. The topology generated by $\mathcal{B}$ is $\mathcal{T}$, but we do not need to go through the extra step of taking finite intersections to form arbitrary open sets in the topology. Sometimes $\mathcal{E}$ is called a *subbase* for the topology $\mathcal{T}(\mathcal{E})$.

Definition E.11 (Locally Convex). If X is a vector space that has a topology $\mathcal{T}$, then we say that X is *locally convex* if there exists a base $\mathcal{B}$ for the topology that consists of convex sets.

For example, if X is a normed linear space, then X is locally convex since each open ball in X is convex. However, as $L^p(E)$ with $p < 1$ illustrates, a metric linear space need not be locally convex in general.

E.2.2 Topological Vector Spaces

A topological vector space is a vector space that has a topology such that the operations of vector addition and scalar multiplication are continuous. In order to define this precisely, the reader should recall the definition of the topology on the product space $X \times X$ as given in Section A.6.

Definition E.12 (Topological Vector Space). A *topological vector space* (TVS) is a vector space X together with a topology $\mathcal{T}$ such that

(a) $(x, y) \mapsto x + y$ is a continuous map of $X \times X$ into X, and
(b) $(c, x) \mapsto cx$ is a continuous map of $\mathbb{C} \times X$ into X.

By Exercise A.8, every normed linear space is a locally convex topological vector space.

Remark E.13. Some authors additionally require in the definition of topological vector space that the topology on X be Hausdorff, and some further require the topology to be locally convex.

Lemma E.14. *If X is a topological vector space, then the topology on X is* translation-invariant*, meaning that if $U \subseteq X$ is open, then $U + x$ is open for every $x \in X$.*

Proof. Suppose $U \subseteq X$ is open. Then, since vector addition is continuous, the inverse image of U under vector addition, which is

$$+^{-1}(U) \;=\; \bigl\{(y, z) \in X \times X : y + z \in U\bigr\},$$

is open in $X \times X$. Exercise A.42 therefore implies that the restriction

$$\bigl\{y \in X : y + z \in U\bigr\}$$

is open in X for each $z \in X$. In particular,

$$U + x \;=\; \bigl\{u + x : u \in U\bigr\} \;=\; \bigl\{y \in X : y + (-x) \in U\bigr\}$$

is open in X. ☐

Additional Problems

E.2. Let X be a normed vector space. Show that the topology induced from the norm is the smallest topology with respect to which X is a topological vector space and $x \mapsto \|x\|$ is continuous.

E.3. Let X be a normed vector space. For each $0 < r < s < \infty$, let A_{rs} be the open annulus $A_{rs} = \{x \in X : r < \|x\| < s\}$ centered at the origin. Let $\mathcal{B}$ consist of $\emptyset$, X, all open balls $B_r(0)$ centered at the origin, and all open annuli A_{rs} centered at the origin. Prove the following facts.

(a) $\mathcal{B}$ is a base for the topology $\mathcal{T}(\mathcal{B})$ generated by $\mathcal{B}$.
(b) $x \mapsto \|x\|$ is continuous with respect to the topology $\mathcal{T}(\mathcal{B})$.
(c) X is not a topological vector space with respect to the topology $\mathcal{T}(\mathcal{B})$.

E.3 Topologies Induced by Families of Seminorms

Our goal in this section is to show that a family of seminorms on a vector space X induces a natural topology on that space, and that X is a locally convex topological vector space with respect to that topology.

E.3.1 Motivation

In order to motivate the construction of the topology associated with a family of seminorms, let us consider the ordinary topology on the Euclidean space $\mathbb{R}^2$. We usually consider this topology to be induced from the Euclidean norm on $\mathbb{R}^2$. Here we will show how the same topology is induced from the two seminorms ρ_1 and ρ_2 on $\mathbb{R}^2$ defined by

$$\rho_1(x_1, x_2) = |x_1| \qquad \text{and} \qquad \rho_2(x_1, x_2) = |x_2|.$$

In analogy with how we create open balls from a norm, define

$$B_r^\alpha(x) \;=\; \{y \in \mathbb{R}^2 : \rho_\alpha(x-y) < r\}, \qquad x \in \mathbb{R}^2,\ r > 0,\ \alpha = 1, 2.$$

These sets are "open strips" instead of open balls, see the illustration in Figure E.2. By taking finite intersections of these strips, we obtain all possible open rectangles $(a, b) \times (c, d)$, and unions of these rectangles exactly give us all the subsets of $\mathbb{R}^2$ that are open with respect to the Euclidean topology. Thus

$$\mathcal{E} \;=\; \bigl\{B_r^\alpha(x) : x \in \mathbb{R}^2,\ r > 0,\ \alpha = 1, 2\bigr\}$$

generates the usual topology on $\mathbb{R}^2$. However, $\mathcal{E}$ is not a base for this topology, since we cannot write an arbitrary open set as a union of open strips. Instead, we have to take one more step: The collection of *finite intersections* of the open strips forms the base. Every open set is a union of finite intersections of open strips. Furthermore, each of these finite intersections of strips is an open rectangle, which is convex, so our base consists of convex sets. Thus, this topology is locally convex. The topology induced from an arbitrary family of seminorms on a vector space will be defined in exactly the same way.

E.3.2 The Topology Associated with a Family of Seminorms

Definition E.15 (Topology Induced from Seminorms). Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X. Then the *αth open strip of radius r centered at $x \in X$* is

$$B_r^\alpha(x) \;=\; \bigl\{y \in X : \rho_\alpha(x-y) < r\bigr\}.$$

Let $\mathcal{E}$ be the collection of all open strips in X:

$$\mathcal{E} \;=\; \bigl\{B_r^\alpha(x) : \alpha \in J,\ r > 0,\ x \in X\bigr\}.$$

The topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ is called the *topology induced by* $\{\rho_\alpha\}_{\alpha \in J}$.

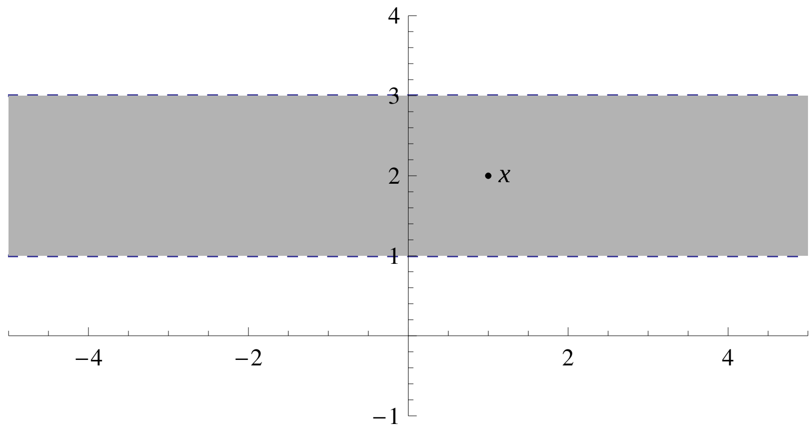

Fig. E.2. The open strip $B_r^2(x)$ for $x = (1,2)$ and $r = 1$.

The fact that ρ_α is a seminorm ensures that each open strip $B_r^\alpha(x)$ is convex. Hence all finite intersections of open strips will also be convex.

One base for the topology generated by the open strips is the collection of all possible finite intersections of open strips. However, it is usually more notationally convenient to use the somewhat smaller base consisting of of finite intersections of strips that are all centered at the same point and have the same radius.

Theorem E.16. *Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X. Then*

$$\mathcal{B} \;=\; \left\{ \bigcap_{j=1}^n B_r^{\alpha_j}(x) : n \in \mathbb{N},\ \alpha_j \in J,\ r > 0,\ x \in X \right\}$$

forms a base for the topology induced from these seminorms. In fact, if U is open and $x \in U$, then there exists an $r > 0$ and $\alpha_1, \dots, \alpha_n \in J$ such that

$$\bigcap_{j=1}^n B_r^{\alpha_j}(x) \;\subseteq\; U.$$

Further, every element of $\mathcal{B}$ is convex.

Proof. Suppose that $U \subseteq X$ and $x \in U$. In order to show that $\mathcal{B}$ is a base for the topology, we have to show that there exists some set $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By the characterization of the generated topology given in Exercise A.39, U is a union of finite intersections of elements of $\mathcal{E}$. Hence we have

$$x \in \bigcap_{j=1}^n B_{r_j}^{\alpha_j}(x_j)$$

for some $n > 0$, $\alpha_j \in J$, $r_j > 0$, and $x_j \in X$. Then $x \in B_{r_j}^{\alpha_j}(x_j)$, so, by definition, $\rho_{\alpha_j}(x - x_j) < r_j$ for each j. Therefore, if we set

$$r = \min\{r_j - \rho_{\alpha_j}(x - x_j) : j = 1, \dots, n\},$$

then we have $B_r^{\alpha_j}(x) \subseteq B_{r_j}^{\alpha_j}(x_j)$ for each $j = 1, \dots, n$. Hence

$$B = \bigcap_{j=1}^{n} B_r^{\alpha_j}(x) \in \mathcal{B},$$

and we have $x \in B \subseteq U$. □

Note that even if there are infinitely many seminorms in our family, when constructing the base we only intersect finitely many strips at a time.

Fortunately, the topology induced by a family of seminorms is always locally convex. Unfortunately, it need not be Hausdorff in general.

Exercise E.17. Let $\{\rho_\alpha\}_{\alpha \in J}$ be a family of seminorms on a vector space X. Show that the induced topology on X is Hausdorff if and only if

$$\rho_\alpha(x) = 0 \text{ for all } \alpha \in J \quad \Longleftrightarrow \quad x = 0.$$

Hausdorffness is an important property because without it the limit of a sequence need not be unique (see Problem A.16). Hence in almost every practical circumstance we require the topology to be Hausdorff. If any one of the seminorms in our family is a norm, then the corresponding topology is automatically Hausdorff (for example, this is the case for $C_b^\infty(\mathbb{R})$, see Example E.3). On the other hand, the topology can be Hausdorff even if no individual seminorm is a norm (consider $L^1_{\mathrm{loc}}(\mathbb{R})$ in Example E.5).

E.3.3 The Convergence Criterion

The meaning of convergence with respect to a net in an arbitrary topological space X was given in Definition A.45. Specifically, a net $\{x_i\}_{i \in I}$ converges to $x \in X$ if for any open neighborhood U of x there exists $i_0 \in I$ such that

$$i \geq i_0 \implies x_i \in U.$$

In this case we write $x_i \to x$.

When the topology is induced from a family of seminorms, we can reformulate the meaning of convergence directly in terms of the seminorms instead of open neighborhoods. Since we are dealing with arbitrary collections of seminorms at this point, we must still deal with convergence in terms of nets rather than ordinary sequences, but even so the fact that the seminorms are real-valued allows a certain amount of notational simplication. Specifically, given a net $\{x_i\}_{i \in I}$ in X and given a seminorm ρ on X, since the open intervals form a base for the topology on $\mathbb{R}$, we have that $\rho(x_i) \to 0$ in $\mathbb{R}$ if and only if for every $\varepsilon > 0$ there exist an $i_0 \in I$ such that

$$i \geq i_0 \implies \rho(x_i) < \varepsilon.$$

The next theorem shows that convergence with respect to the topology induced from a family of seminorms is exactly what we expect it should be, namely, simultaneous convergence with respect to each individual seminorm.

Theorem E.18. *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$. Then given any net $\{x_i\}_{i\in i}$ and any $x \in X$, we have*

$$x_i \to x \quad \iff \quad \forall\, \alpha \in J,\ \rho_\alpha(x - x_i) \to 0.$$

Proof. $\Rightarrow$. Suppose that $x_i \to x$, and fix any $\alpha \in J$ and $\varepsilon > 0$. Then $B_\varepsilon^\alpha(x)$ is an open neighborhood of x, so by definition of convergence with respect to a net, there exists an $i_0 \in I$ such that

$$i \geq i_0 \quad \Longrightarrow \quad x_i \in B_\varepsilon^\alpha(x).$$

Therefore, for all $i \geq i_0$ we have $\rho_\alpha(x - x_i) < \varepsilon$, so $\rho_\alpha(x - x_i) \to 0$.

$\Leftarrow$. Suppose that $\rho_\alpha(x - x_i) \to 0$ for every $\alpha \in J$, and let U be any open neighborhood of x. Then by Theorem E.16, we can find an $r > 0$ and finitely many $\alpha_1, \dots, \alpha_n \in J$ such that

$$x \ \in\ \bigcap_{j=1}^{n} B_r^{\alpha_j}(x) \ \subseteq\ U.$$

Now, given any $j = 1, \dots, n$ we have $\rho_{\alpha_j}(x - x_i) \to 0$. Hence, for each j we can find a $k_j \in I$ such that

$$i \geq k_j \quad \Longrightarrow \quad \rho_{\alpha_j}(x - x_i) < r.$$

Since I is a directed set, there exists some $i_0 \in I$ such that $i_0 \geq k_j$ for $j = 1, \dots, n$. Thus, for all $i \geq i_0$ we have $\rho_{\alpha_j}(x - x_i) < r$ for each $j = 1, \dots, n$, so

$$x_i \ \in\ \bigcap_{j=1}^{n} B_r^{\alpha_j}(x) \ \subseteq\ U, \qquad i \geq i_0.$$

Hence $x_i \to x$. □

By combining Theorem E.18 with Lemma A.54, we obtain a criterion for continuity in terms of the seminorms.

Corollary E.19. *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$, let Y be any topological space, and fix $x \in X$. Then the following two statements are equivalent.*

(a) *$f\colon X \to Y$ is continuous at x.*

(b) *For any net $\{x_i\}_{i\in I}$ in X,*

$$\rho_\alpha(x - x_i) \to 0 \ \text{ for each } \alpha \in J \quad \Longrightarrow \quad f(x_i) \to f(x) \ \text{ in } Y.$$

In particular, if $\mu\colon X \to \mathbb{C}$ is a linear functional, then μ is continuous if and only if for each net $\{x_i\}_{i\in I}$ in X we have

$$\rho_\alpha(x_i) \to 0 \ \text{ for each } \alpha \in J \quad \Longrightarrow \quad \langle x_i, \mu\rangle \to 0.$$

The *dual space* X^* of X is the space of all continuous linear functionals on X.

Remark E.20. Because of the Reverse Triangle Inequality, $\rho_\alpha(x - x_i) \to 0$ implies $\rho_\alpha(x_i) \to \rho_\alpha(x)$. Hence each seminorm ρ_α is continuous with respect to the induced topology.

E.3.4 Continuity of the Vector Space Operations

Now we can show that a vector space with a topology induced from a family of seminorms is a topological vector space.

Theorem E.21. *If X is a vector space whose topology is is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha \in J}$, then X is a locally convex topological vector space.*

Proof. We have already seen that there is a base for the topology that consists of convex open sets, so we just have to show that vector addition and scalar multiplication are continuous with respect to this topology.

Suppose that $\{(c_i, x_i)\}_{i \in I}$ is any net in $\mathbb{C} \times X$, and that $(c_i, x_i) \to (c, x)$ with respect to the product topology on $\mathbb{C} \times X$. By Problem A.20, this is equivalent to assuming that $c_i \to c$ in $\mathbb{C}$ and $x_i \to x$ in X. Fix any $\alpha \in J$ and any $\varepsilon > 0$. Suppose that $\rho_\alpha(x) \neq 0$. Since $\rho_\alpha(x - x_i) \to 0$, there exist i_1, $i_2 \in I$ such that

$$i \geq i_1 \quad \Longrightarrow \quad |c - c_i| < \min\left\{\frac{\varepsilon}{2\,\rho_\alpha(x)},\, 1\right\}.$$

and

$$i \geq i_2 \quad \Longrightarrow \quad \rho_\alpha(x - x_i) < \frac{\varepsilon}{2\,(|c| + 1)}$$

By definition of directed set, there exists some $i_0 \geq i_1, i_2$, so both of these inequalities hold for $i \geq i_0$. In particular, $\{c_i\}_{i \geq i_0}$ is a bounded sequence, with $|c_i| < |c| + 1$ for all $i \geq i_0$. Hence, for $i \geq i_0$ we have

$$\begin{aligned} \rho_\alpha(cx - c_i x_i) &\leq \rho_\alpha(cx - c_i x) + \rho_\alpha(c_i x - c_i x_i) \\ &= |c - c_i|\,\rho_\alpha(x) + |c_i|\,\rho_\alpha(x - x_i) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon. \end{aligned}$$

If $\rho_\alpha(x) = 0$ then we similarly obtain $\rho_\alpha(cx - c_i x_i) < \varepsilon/2$ for $i \geq i_0$. Thus we have $\rho_\alpha(cx - c_i x_i) \to 0$. Since this is true for every α, Corollary E.19 implies that $c_i x_i \to cx$.

Exercise: Finish the proof by showing that vector addition is continuous. □

E.3.5 Continuity Equals Boundedness

For linear maps on normed vector spaces, Theorem C.6 tells us that continuity is equivalent to boundedness. We will now prove an analogous result for operators on vector spaces whose topogies are induced from families of seminorms. In the statement of the following result, it is perhaps surprising at first glance that "boundedness" of a given operator is completely determined by a fixed *finite* subcollection of the seminorms. This is a reflection of the construction of the topology, and specifically of the fact that a base for the topology is obtained by intersecting only finitely many open strips at a time.

Theorem E.22 (Continuity Equals Boundedness). *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in I}$. Let Y be a vector space whose topology is induced from a family of seminorms $\{q_\beta\}_{\beta\in J}$. If $L\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *L is continuous.*

(b) *For each $\beta \in J$, there exist $N \in \mathbb{N}$, $\alpha_1, \dots, \alpha_N \in I$, and $C > 0$ (all depending on β) such that*

$$q_\beta(Lx) \;\le\; C \sum_{j=1}^{N} \rho_{\alpha_j}(x), \qquad x \in X. \tag{E.1}$$

Proof. (a) $\Rightarrow$ (b). Assume that L is continuous, and fix $\beta \in J$. Since q_β is continuous, so is $q_\beta \circ L$. Hence

$$(q_\beta \circ L)^{-1}(-1,1) \;=\; \{x \in X : q_\beta(Lx) < 1\}$$

is open in X. Further, this set contains $x = 0$, so there must exist a base set

$$B \;=\; \bigcap_{j=1}^{N} B_r^{\alpha_j}(0)$$

such that

$$B \;\subseteq\; \{x \in X : q_\beta(Lx) < 1\}. \tag{E.2}$$

We will show that equation (E.1) is satisfied with $C = 2/r$. To see this, fix any $x \in X$, and set

$$\delta \;=\; \sum_{j=1}^{N} \rho_{\alpha_j}(x).$$

Case 1: $\delta = 0$. In this case, given any $\lambda > 0$ we have

$$\rho_{\alpha_j}(\lambda x) \;=\; \lambda\, \rho_{\alpha_j}(x) \;=\; 0, \qquad j = 1, \dots, N.$$

Hence $\lambda x \in B_r^{\alpha_j}(x)$ for each $j = 1, \dots, N$, so $\lambda x \in B$. In light of the inclusion in equation (E.2), we therefore have for every $\lambda > 0$ that

$$\lambda\, q_\beta(Lx) \;=\; q_\beta(L(\lambda x)) \;<\; 1.$$

Therefore $q_\beta(Lx) = 0$. Hence in this case the inequality (E.1) is trivially satisfied.

Case 2: $\delta > 0$. In this case we have for each $j = 1, \dots, N$ that

$$\rho_{\alpha_j}\Big(\frac{rx}{2\delta}\Big) \;=\; \frac{r}{2\delta}\,\rho_{\alpha_j}(x) \;\le\; \frac{r}{2} \;<\; r,$$

so $\frac{rx}{2\delta} \in B$. Considering equation (E.2), we therefore have

$$\frac{r}{2\delta}\,q_\beta(Lx) \;=\; q_\beta\Big(L\Big(\frac{rx}{2\delta}\Big)\Big) \;<\; 1.$$

Hence

$$q_\beta(Lx) \;<\; \frac{2\delta}{r} \;=\; \frac{2}{r}\sum_{j=1}^{N}\rho_{\alpha_j}(x),$$

which is the desired inequality.

(b) ⇒ (a). Suppose that statement (b) holds. Let $\{x_i\}_{i\in A}$ be any net in X, and suppose that $x_i \to x$ in X. Then for each $\alpha \in I$, we have by Corollary E.19 that $\rho_\alpha(x - x_i) \to 0$ for each $\alpha \in I$. Equation (E.1) therefore implies that $q_\beta(Lx - Lx_i) \to 0$ for each $\beta \in J$. Using Corollary E.19 again, this says that $Lx_i \to Lx$ in Y, so L is continuous. □

Often, the space Y will be a normed space. In this case, the statement of Theorem E.22 simplifies as follows.

Corollary E.23. *Let X be a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$, and let Y be a normed linear space. If $L\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *L is continuous.*

(b) *There exist $N \in \mathbb{N}$, $\alpha_1, \dots, \alpha_N \in J$, and $C > 0$ such that*

$$\|Lx\| \;\le\; C\sum_{j=1}^{N}\rho_{\alpha_j}(x), \qquad x \in X. \tag{E.3}$$

Thus, to show that a given linear operator $L\colon X \to Y$ is continuous, we need only find *finitely many* seminorms such that the boundedness condition in equation (E.3) holds.

E.4 Topologies Induced by Countable Families of Seminorms

In this section, we will show that if a Hausdorff topology is induced from a *countable* collection of seminorms, then it is metrizable. In particular, we can define continuity of operators on such a space using convergence of ordinary sequences instead of nets.

E.4.1 Metrizing the Topology

Exercise E.24. Let X be a vector space whose topology is induced from a countable family of seminorms $\{\rho_n\}_{n\in\mathbb{N}}$, and assume that X is Hausdorff. Prove the following statements.

(a) The function

$$\mathrm{d}(x,y) \;=\; \sum_{n=1}^{\infty} 2^{-n}\,\frac{\rho_n(x-y)}{1+\rho_n(x-y)}$$

defines a metric on X.

(b) The metric d generates the same topology as the family of seminorms $\{\rho_n\}_{n\in\mathbb{N}}$.

(c) The metric d is translation-invariant, i.e.,

$$\mathrm{d}(x+z,y+z) \;=\; \mathrm{d}(x,y), \qquad x,y,z\in X.$$

Since the metric defines the same topology as the seminorms, they define the same convergence criteria, i.e., given a sequence $\{x_k\}_{k\in\mathbb{N}}$ in X and given $x\in X$, we have

$$\mathrm{d}(x,x_k)\to 0 \quad\Longleftrightarrow\quad \forall\, n\in\mathbb{N},\ \rho_n(x-x_k)\to 0.$$

Definition E.25 (Fréchet Space). Let X be a Hausdorff vector space whose topology is induced from a countable family of seminorms $\{\rho_n\}_{n\in\mathbb{N}}$. If X is complete with respect to the metric constructed in Exercise E.24, then we call X a *Fréchet space*.

Exercise E.26. Let X be as above, and suppose that a sequence $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy with respect to the metric d. Show that $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy with respect to each individual seminorm ρ_n.

Exercise E.27. Show that the spaces $C_b^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, $L^1_{\mathrm{loc}}(\mathbb{R})$, and $C^\infty(\mathbb{R})$ considered in Examples E.3–E.6 are all Fréchet spaces.