## E.6 The Weak and Weak* Topologies on a Normed Linear Space

The weak topology on a normed space and the weak* topology on the dual of a normed space were introduced in Examples E.7 and E.8. We will study these topologies more closely in this section. They are specific examples of generic "weak topologies" determined by the requirement that a given class of mappings $f_\alpha : X \to Y_\alpha$ be continuous. The "weak topology" corresponding to such a class is the weakest (smallest) topology such that each map $f_\alpha$ is continuous. Another example of such a weak topology is the product topology, which is defined in Section E.7.

### E.6.1 The Weak Topology

Let $X$ be a normed space. The topology induced from the norm on $X$ is called the *strong* or *norm topology* on $X$.

For each $\mu \in X^*$, the functional

$$\rho_\mu(x) \;=\; |\langle x, \mu\rangle|, \qquad x \in X,$$

is a seminorm on $X$. The topology induced by the family of seminorms $\{\rho_\mu\}_{\mu\in X^*}$ is the weak topology on $X$, denoted

$$\sigma(X, X^*).$$

By Theorem E.16, $X$ is a locally convex topological vector space with respect to the weak topology. If $\rho_\mu(x) = 0$ for every $\mu \in X^*$, then, by the Hahn–Banach Theorem,

$$\|x\| \;=\; \sup_{\|\mu\|=1} |\langle x, \mu\rangle| \;=\; \sup_{\|\mu\|=1} \rho_\mu(x) \;=\; 0.$$

Hence $x = 0$, so the weak topology is Hausdorff (see Exercise E.17). If we let

$$B_r^\mu(x) \;=\; \bigl\{y \in X : \rho_\mu(x-y) < r\bigr\} \;=\; \bigl\{y \in X : |\langle x-y, \mu\rangle| < r\bigr\}$$

denote the open strips determined by these seminorms, then a base for the topology $\sigma(X, X^*)$ is

$$\mathcal{B} \;=\; \left\{\bigcap_{j=1}^{n} B_r^{\mu_j}(x) : n \in \mathbb{N},\; \mu_j \in X^*,\; r > 0,\; x \in X\right\}.$$

Convergence with respect to this topology in called weak convergence in $X$, denoted $x_i \xrightarrow{\mathrm{w}} x$.

Suppose that $\mu \in X^*$ and that $\{x_i\}_{i\in I}$ is a net in $X$ such that $x_i \xrightarrow{\mathrm{w}} 0$. By Remark E.20, $\rho_\mu$ is continuous with respect to the weak topology, so

$$|\langle x_i, \mu\rangle| \;=\; \rho_\mu(x_i) \;\to\; 0.$$

Thus $\mu$ is continuous with respect to the weak topology as well. The next exercise shows that the weak topology is the *smallest* topology with respect to which each $\mu \in X^*$ is continuous.

**Exercise E.42.** Suppose that $X$ is a normed space, and that $\mathcal{T}$ is a topology on $X$ such that each $\mu \in X^*$ is continuous with respect to $\mathcal{T}$. Show that $\sigma(X, X^*) \subseteq \mathcal{T}$.

Even though the weak topology is weaker than the norm topology, there are a number of situations where we have the surprise that a "weak property" implies a "strong property." For example, Problem E.5 shows that every weakly closed subset of a normed space is strongly closed, and vice versa.

### E.6.2 The Weak* Topology

Let $X$ be a normed space. For each $x \in X$, the functional

$$\rho_x(\mu) \;=\; |\langle x, \mu \rangle|, \qquad \mu \in X^*,$$

is a seminorm on $X^*$. The topology induced by the family of seminorms $\{\rho_x\}_{x \in X}$ is the weak topology on $X^*$, denoted

$$\sigma(X^*, X).$$

By Theorem E.16, $X^*$ is a locally convex topological vector space with respect to the weak* topology. Further, if $\rho_x(\mu) = 0$ for every $x \in X$ then, by definition of the operator norm,

$$\|\mu\| \;=\; \sup_{\|x\|=1} |\langle x, \mu \rangle| \;=\; \sup_{\|x\|=1} \rho_x(\mu) \;=\; 0.$$

Hence $\mu = 0$, so the weak* topology is Hausdorff. If we let

$$B_r^x(\mu) \;=\; \bigl\{\nu \in X^* : \rho_x(\mu - \nu) < r\bigr\} \;=\; \bigl\{\nu \in X^* : |\langle x, \mu - \nu \rangle| < r\bigr\}$$

denote the open strips determined by these seminorms, then a base for the topology $\sigma(X^*, X)$ is

$$\mathcal{B} \;=\; \left\{\bigcap_{j=1}^{n} B_r^{x_j}(\mu) : n \in \mathbb{N},\ x_j \in X,\ r > 0,\ \mu \in X^*\right\}.$$

Convergence with respect to this topology in called weak* convergence in $X^*$, denoted $x_i \xrightarrow{\text{w}^*} x$.

Each vector $x \in X$ is identified with the functional $\widehat{x} \in X^{**}$ defined by

$$\langle \mu, \widehat{x}\, \rangle \;=\; \langle x, \mu \rangle, \qquad \mu \in X^*.$$

Suppose that $x \in X$ and that $\{\mu_i\}_{i \in I}$ is a net in $X^*$ such that $\mu_i \xrightarrow{\text{w}^*} 0$. Then

$$|\langle \mu_i, \widehat{x} \rangle| \;=\; |\langle x, \mu_i \rangle| \;=\; \rho_x(\mu_i) \;\to\; 0,$$

so $\widehat{x}$ is continuous with respect to the weak* topology. The next exercise shows that the weak* topology is the smallest topology with respect to which $\widehat{x}$ is continuous for each $x \in X$.

**Exercise E.43.** Suppose that $X$ is a normed space, and that $\mathcal{T}$ is a topology on $X^*$ such that $\widehat{x}$ is continuous with respect to $\mathcal{T}$ for each $x \in X$. Show that $\sigma(X^*, X) \subseteq \mathcal{T}$.

## Additional Problems

**E.4.** Let $X$ be a normed space, and let $\mathcal{T}$ be the strong topology on $X$.

(a) Show directly that $\sigma(X, X^*) \subseteq \mathcal{T}$.

(b) Use the fact that each $\mu \in X^*$ is continuous (by definition) in the strong topology to show that $\sigma(X, X^*) \subseteq \mathcal{T}$.

**E.5.** Let $X$ be a normed space and $S$ a subspace of $X$. Prove that the following statements are equivalent.

(a) $S$ is strongly closed (i.e., closed with respect to the norm topology).

(b) $S$ is weakly closed (i.e., closed with respect to the weak topology).

**E.6.** Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis for a Hilbert space $H$.

(a) Show that $e_n \xrightarrow{\text{w}} 0$.

(b) Show that $n^{1/2} e_n$ does not converge weakly to 0.

(c) Show that 0 belongs to the weak closure of $\{n^{1/2} e_n\}_{n \in \mathbb{N}}$, i.e., 0 is an accumulation point of this set with respect to the weak topology on $H$.