2. The F.T. & $L^2(\mathbb{R})$

2.1 Definition & Basic Properties

We have defined the F.T. on $L^1(\mathbb{R})$. But we also know that $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ into itself, & $\mathcal{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \le p < \infty$.

In particular, $\mathcal{F}$ maps a dense subset of $L^2(\mathbb{R})$ to a dense subset of $L^2(\mathbb{R})$.

If $\mathcal{F}$ is bounded w.r.t. the L^2-norm, then it will have an extension to all of $L^2(\mathbb{R})$.

<u>Exercise</u>

Prove that $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for each $1 \le p \le 2$.

Proposition Isometry of the F.T. on $\mathcal{S}(\mathbb{R})$

$$\|\hat{f}\|_2 = \|f\|_2 \quad \forall f \in \mathcal{S}(\mathbb{R}).$$

Proof:

Assume $f \in \mathcal{S}(\mathbb{R})$. Then $\hat{f} \in \mathcal{S}(\mathbb{R})$, so $f, \hat{f} \in L^2(\mathbb{R})$.
Define the involution of f to be

$$\tilde{f}(x) = \overline{f(-x)}.$$

Define
$$g = f * \tilde{f}.$$

Since $f, \tilde{f} \in L^1(\mathbb{R})$ we have $g \in L^1(\mathbb{R})$, so $\hat{g} \in C_0(\mathbb{R})$.

We have

$$\begin{aligned} g(0) &= \int f(y)\, \tilde{f}(0-y)\, dy \\ &= \int f(y)\, \overline{f(y)}\, dy \\ &= \|f\|_2^2 . \end{aligned}$$

Also,

$$\begin{aligned} \hat{g}(\xi) &= \hat{f}(\xi)\, \hat{\tilde{f}}(\xi) \\ &= \hat{f}(\xi)\, \overline{\hat{f}(\xi)} \qquad \text{(Exercise)} \\ &= |\hat{f}(\xi)|^2 \\ &\in L^1(\mathbb{R}). \end{aligned}$$

Therefore the inversion formula applies, and

$g = \hat{g}^{\vee}$ everywhere since they are continuous.

Hence

$$
\begin{aligned}
\|f\|_2^2 &= g(0) \\
&= \hat{g}^{\vee}(0) \\
&= \int \hat{g}(\xi)\, d\xi \\
&= \int |\hat{f}(\xi)|^2\, d\xi \\
&= \|\hat{f}\|_2^2 . \qquad \blacksquare
\end{aligned}
$$

<u>Exercise</u>

We didn't really need to use the full power of the assumption $f \in \mathcal{S}(\mathbb{R})$. Find weaker hypotheses under which the proof will still work.

Here is another proof that the F.T. is an isometry on a dense subset of $L^2(\mathbb{R})$.

<u>Proposition</u>

$$\|f\|_2 = \|\hat{f}\|_2 \qquad \forall f \in C_c(\mathbb{R}).$$

<u>Proof:</u>

Fix $f \in C_c(\mathbb{R})$ & set $g = f * \tilde{f} \in C_c(\mathbb{R}) \subseteq L^1(\mathbb{R})$.

As before, $g(0) = \|f\|_2^2$ & $\hat{g}(\xi) = |\hat{f}(\xi)|^2$.

Let ω be the Fejer kernel.

Exercise: Show that $g \in C_c(\mathbb{R})$, $\omega \in C_0(\mathbb{R})$ implies

$$g * \omega_\lambda(x) \longrightarrow g(x) \text{ pointwise as } \lambda \to \infty.$$

Therefore $g * \omega_\lambda(0) \longrightarrow g(0) = \|f\|_2^2$.

Also,

$$g * \omega_\lambda(0) = \int_{-\lambda}^{\lambda} \hat{g}(\xi)\left(1 - \frac{|\xi|}{\lambda}\right) \cancelto{1}{e^{2\pi i \xi 0}} \, d\xi$$

$$= \int_{-\lambda}^{\lambda} |\hat{f}(\xi)|^2 \left(1 - \frac{|\xi|}{\lambda}\right) d\xi$$

$$\longrightarrow \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = \|\hat{f}\|_2^2.$$

by the Monotone Convergence Theorem.

Consequently, $\|\hat{f}\|_2^2 = \|f\|_2^2$ is finite. $\blacksquare$

<u>Definition</u> <u>F.T. on $L^2(\mathbb{R})$</u>

Given $f \in L^2(\mathbb{R})$, let $f_n \in C_c(\mathbb{R})$ be s.t.

$f_n \longrightarrow f$ in L^2- norm. Then

$$\|\hat{f}_m - \hat{f}_n\|_2 = \|(f_m - f_n)^\wedge\|_2 = \|f_m - f_n\|_2,$$

so $\{\hat{f}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R})$,

and therefore converges to some function $g \in L^2(\mathbb{R})$.

The Fourier Transform of f is defined to be

$$\hat{f} = g.$$

Exercises

(a) Prove that the definition of the F.T. is well-defined, i.e., it is independent of the choice of sequence $f_n \in C_c(\mathbb{R})$ with $f_n \to f$. Prove $\mathcal{F}$ is linear.

(b) Prove Plancherel's Theorem:

$$\|\hat{f}\|_2 = \|f\|_2 \quad \forall f \in L^2(\mathbb{R}).$$

(c) Prove that if $\{f_n\}$ is any sequence of functions in $L^2(\mathbb{R})$ s.t. $f_n \to f$, then $\hat{f}_n \to \hat{f}$ in L^2-norm.

(d) Prove Parseval's Theorem:

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \quad \forall f, g \in L^2(\mathbb{R}).$$

(e) Prove that $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a unitary map of $L^2(\mathbb{R})$ onto itself.

Exercise

Suppose X, Y are Banach spaces, $D \subseteq X$ is dense, $T : D \to Y$ is linear & continuous. Prove that T has a unique extension to a bounded linear map $\tilde{T} : X \to Y$, and furthermore $\|\tilde{T}\| = \|T\|$.

Exercise

Prove that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the L^2-definition of the F.T. of f coincides with the L^1-definition, i.e., the two definitions of $\hat{f}$ are equal a.e.

Exercise

Suppose $f \in L^1(\mathbb{R})$ is s.t. $\hat{f} \in L^2(\mathbb{R})$.
Prove that $f \in L^2(\mathbb{R})$.
Warning: Be careful. Hint: Approximate identities.

Exercise

Let $f \in L^2(\mathbb{R})$ be given. Prove that

$$\{T_a f\}_{a \in \mathbb{R}} \text{ is complete in } L^2(\mathbb{R}) \iff \hat{f}(\xi) \neq 0 \text{ a.e.}$$

Remarks

(a) If $f \in L^2(\mathbb{R})$ then $\hat{f}$ is only defined a.e. $\hat{f}$ can be any function in $L^2(\mathbb{R})$ & need not be continuous.

(b) If $f \in L^2(\mathbb{R})$ then $f_n = f \cdot \chi_{[-n,n]} \in L^1 \cap L^2$

and $f_n \to f$ in L^2-norm. Hence $\hat{f}_n \to \hat{f}$ in L^2, ie,

$$\hat{f} = \lim_{n\to\infty} \hat{f}_n = \lim_{n\to\infty} \int_{-n}^{n} f(x)\, e^{-2\pi i \xi x}\, dx$$

but note that this is a limit in L^2-norm, not a pointwise limit.

(c) We often abuse notation and write

$$\hat{f}(\xi) = \int f(x)\, e^{-2\pi i \xi x}\, dx$$

even for $f \in L^2(\mathbb{R})$, but note that the integral is defined only for $f \in L^1(\mathbb{R})$.

(d) The Riemann-Lebesgue lemma fails on $L^2(\mathbb{R})$!

$$f \in L^2(\mathbb{R}) \not\Rightarrow \lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$$

Exercise

Similarly define the inverse F.T. $\check{f}$ for $f \in L^2(\mathbb{R})$.
Prove that

$$f = \hat{f}^{\vee} = \check{f}^{\wedge} \quad \forall f \in L^2(\mathbb{R}).$$

Exercise

Prove that $\quad \hat{d}_{2\pi T} = \chi_{[-T,T]}$.

In Chapter 7

Exercise

Prove that $C_c^2(\mathbb{R}) \subseteq A(\mathbb{R})$.

That is, show that if $f \in C^c_2(\mathbb{R})$, i.e.,

2 continuous derivatives with compact support,

then $\hat{f} \in L^1(\mathbb{R})$.

Exercise

a. Prove there is no $f \in L^1(\mathbb{R}) \setminus \{0\}$ s.t. $f = f * f$.

b. Prove there exist $f \in L^2(\mathbb{R}) \setminus \{0\}$ s.t. $f = f * f$.

Exercise

Prove that if $f \in L^1(\mathbb{R})$ & $g \in L^2(\mathbb{R})$ then $(f*g)^\wedge = \hat{f}\hat{g}$
(note $f*g \in L^2(\mathbb{R})$).

Exercise

Show that if $f, g \in L^2(\mathbb{R})$ then $(fg)^\wedge = \hat{f} * \hat{g}$.

Note that $fg \in L^1(\mathbb{R})$ & $\hat{f} * \hat{g} \in L^2 * L^2 \subseteq L^\infty$.

Exercise

a. Show $\exists f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $f \neq 0$ s.t.

$$f * d_{2\pi\Omega} = 0$$

Hint: $d_{2\pi\Omega} \in L^2(\mathbb{R})$ & $\hat{d}_{2\pi\Omega} = \chi_{[-\Omega,\Omega]}$.

b. Prove that if $a \neq 0$ & $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ satisfies

$$f * d_{2\pi\Omega} = a f \cdot \chi_{[-T,T]}$$

then $f = 0$ on $(-T,T)$ & $\hat{f} = 0$ on $(-\Omega,\Omega)$.

Fact (not obvious): This implies $f = 0$.

Compare this fact to Paley-Wiener Theorem & the Uncertainty Principles we will prove later.

Exercise

Compute $\displaystyle\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$.

Hint: Write the integral as an inner product. Recall the Poisson kernel $p(x) = \dfrac{1}{\pi(1+x^2)}$ and its F.T. $\hat{p}(\xi) = e^{-2\pi|\xi|}$. Use the Parseval Formula.

Exercise

a. Show that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$.

Hint: Plancherel.

b. Although $\frac{\sin t}{t}$ is not integrable, the improper Riemann integral $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ does exist.

Prove that

$$\int_0^b \frac{\sin^2 t}{t^2}\,dt = -\frac{\sin^2 b}{b} + \int_0^b \frac{\sin 2t}{t}\,dt.$$

Use this to show that

$$\int_0^\infty \frac{\sin t}{t}\,dt = \int_0^\infty \frac{\sin^2 t}{t^2}\,dt.$$

Definition

A subset $M \subseteq L^2(\mathbb{R})$ is *translation-invariant* if

$$f \in M,\ a \in \mathbb{R} \implies T_a f \in M.$$

Exercise

Prove that the following two statements are equivalent:

(a) M is a closed, translation-invariant subspace of $L^2(\mathbb{R})$.

(b) $M = L^2_\Omega(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp}(\hat{f}) \subseteq \Omega\}$

for some $\Omega \subseteq \mathbb{R}$.

Hint for (a) $\Rightarrow$ (b). Let $\{f_n\}_{n=1}^{\infty}$ be an ONB for M.

See Dym & McKean, Section 2.5

2.2 The Hermite Functions

& Folland, Real Analysis, p.256

In this section we will construct an ONB $\{H_n\}_{n=0}^{\infty}$ for $L^2(\mathbb{R})$ consisting of eigenfunctions of the F.T. $\mathcal{F}$.

Exercise A

a. Let $R: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the reflection $Rf(x) = f(-x)$. Prove that

$$\mathcal{F}^2 = R \qquad \& \qquad \mathcal{F}^4 = I.$$

b. Prove that the only possible eigenvalues of $\mathcal{F}$ are $1, i, -1, -i$.

Remark

An earlier exercise proved that $\hat{\phi} = \phi$ when $\phi(x) = e^{-\pi x^2}$ is the Gaussian. Hence ϕ is a 1-eigenvector for $\mathcal{F}$.

Definition

The n^{th} *Hermite Function* is

$$H_n(x) = e^{\pi x^2} D_n e^{-2\pi x^2}, \qquad n \geq 0,$$

where $D_n = \dfrac{d^n}{dx^n}$. Observe that

$H_0 = \phi$, $\hat{H}_0 = H_0$.

Exercise B: This will show that H_n is an eigenvector for $\mathcal{F}$ with eigenvalue $(-i)^n$.

a. Prove that

$$(*) \quad H_{n+1}(x) = H_n'(x) - 2\pi x H_n(x), \qquad n \geq 0.$$

b. Prove that

$$\hat{H}_{n+1}(\xi) = -i \hat{H}_n'(\xi) + 2\pi i \xi \hat{H}_n(\xi)$$

c. Prove that if $G_n = (-i)^n H_n$ then

$$G_{n+1}(x) = -i G_n'(x) + 2\pi i x G_n(x).$$

d. Use $\hat{H}_0 = H_0$ together with parts b & c to show that

$$\hat{H}_n = G_n = (-i)^n H_n$$

Remark: $Hf(x) = -f''(x) + 4\pi x^2 f(x)$ is the <u>Hermite operator</u>

<u>Exercise F:</u> This will show $\{H_n\}_{n=0}^{\infty}$ is an orthogonal basis for $L^2(\mathbb{R})$.

a. Prove that $T^* H_n = 4\pi n H_{n-1}$ (note H_{n-1}, not H_{n+1}).

Hint: Expand H_{n+1} in two ways: first using Exercise B, & then using Exercise D.

b. Show that $K = TT^*$ is given by

$$Kf(x) = -f''(x) + (4\pi x^2 - 2\pi) f(x)$$

c. Show that H_n is an eigenfunction of K.

d. Show that $\{H_n\}_{n=0}^{\infty}$ is an orthogonal sequence.

Hint: What do you know about eigenvectors of a self-adjoint operator?

d. Show that $\{H_n\}_{n=0}^{\infty}$ is complete.

Hints: Suppose $f \in L^2(\mathbb{R})$, $f \perp H_n$ $\forall n \geq 0$. Write

$$(f H_0)^\wedge(\xi) = \int f(x) e^{-\pi x^2} e^{-2\pi i \xi x}\, dx.$$

Expand the exponential in a Taylor series.

Exercise C

Use equation (*) to show that

$$H_n(x) = p_n(x)\, e^{-\pi x^2}, \qquad n \geq 0,$$

where p_n is a polynomial of degree n (called the n^{th} Hermite polynomial).

Exercise D

Prove that if f is an n-times differentiable function, then

$$D^n\big(x f(x)\big) = x D^n f(x) + n D^{n-1} f(x)$$

Exercise E

Define $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ by

$$T f(x) = f'(x) - 2\pi x f(x).$$

Exercise B showed that $H_{n+1} = T H_n$.

Prove that the adjoint of T (as an unbounded operator on $L^2(\mathbb{R})$) is

$$T^* f(x) = -f'(x) - 2\pi x f(x),$$

i.e., $\langle Tf, g\rangle = \langle f, T^* g\rangle \quad \forall f, g \in \mathcal{S}(\mathbb{R})$.

$$e^{2\pi i \xi x} = \sum_{n=0}^{\infty} \frac{(-2\pi i \xi)^n \, x^n}{n!}$$

Show the integral & sum can be interchanged

<u>Exercise G</u>: Now we normalize the H_n.

a. Show that $\|H_0\|_2^2 = 2^{-1/2}$.

b. Show that $\|H_n\|_2^2 = 4\pi n \, \|H_{n-1}\|_2^2$, $n > 0$.

Hint: Use Exercise F part a.

c. Find $\|H_n\|_2$ explicitly.

d. Set $h_n = H_n / \|H_n\|_2$ & conclude that $\{h_n\}_{n=0}^{\infty}$ is an ONB of $L^2(\mathbb{R})$ consisting of eigenvectors of $\mathcal{F}$. Note further that $h_n \in \mathcal{S}(\mathbb{R})$ $\forall n$; in fact each h_n is ∞-differentiable & has <u>exponential</u> decay at ∞.

Remark: Wiener's definition of the F.T.

Set $h_n = H_n / \|H_n\|_2$, so $\{h_n\}_{n=0}^{\infty}$ is an ONB for $L^2(\mathbb{R})$. Then we can use this ONB to define the F.T., i.e., we set

$$\mathcal{F}f = \hat{f} = \sum_{n=0}^{\infty}{}' (-i)^n \langle f, h_n \rangle h_n, \qquad f \in L^2(\mathbb{R}),$$

where the series converges in L^2-norm.

Exercise

a. Show that $\mathcal{F}$ defined as above is a unitary map of $L^2(\mathbb{R})$ onto itself.

b. Show that this definition of the F.T. on $L^2(\mathbb{R})$ coincides with our previous definition.

Hint: For finite N, $\sum_{n=0}^{N}{}' \langle f, h_n \rangle h_n \in \mathcal{S}(\mathbb{R})$.

2.3 The F.T. on $L^p(\mathbb{R})$, $1 \le p \le 2$

We have shown that

$$\begin{array}{lcccl}
\mathcal{F}: & L^1(\mathbb{R}) & \longrightarrow & L^\infty(\mathbb{R}), & \|\mathcal{F}\| = 1 \\
& \updownarrow p? & & \updownarrow p'? & \\
\mathcal{F}: & L^2(\mathbb{R}) & \longrightarrow & L^2(\mathbb{R}), & \|\mathcal{F}\| = 1 \\
& & & & \text{(in fact, unitary)}
\end{array}$$

Further, $L^1 \cap L^2$ is dense in L^p for $1 \le p \le 2$.

Can we extend $\mathcal{F}$ from $L^1 \cap L^2$ to all of L^p?

Remark:

If $f \in L^p(\mathbb{R})$, $1 \le p \le 2$, then $f = f_1 + f_2$

with $f_1 \in L^1(\mathbb{R})$, $f_2 \in L^2(\mathbb{R})$, e.g.,

$$f_1(x) = \begin{cases} f(x), & |f(x)| > 1 \\ 0, & |f(x)| \le 1 \end{cases} \quad \& \quad f_2 = f - f_1.$$

Hence we can define $\hat{f} = \hat{f}_1 + \hat{f}_2 \in L^\infty + L^2$.

However, this can be improved by using <u>interpolation</u>

theorems

Riesz-Thorin Interpolation Theorem [see J. Duoandikoatxea, Fourier Analysis]

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ be given.

Assume $T: L^{p_0} + L^{p_1} \to L^{q_0} + L^{q_1}$ is linear and set

$$M_0 = \|T\|_{L^{p_0} \to L^{q_0}} \qquad M_1 = \|T\|_{L^{p_1} \to L^{q_1}}.$$

Given $0 < \theta < 1$ define p, q by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then $T: L^p \to L^q$, and

$$\|T\|_{L^p \to L^q} \leq M_0^{1-\theta} M_1^{\theta}$$

Exercise: Use Riesz-Thorin to prove Young's convolution inequalities.

By applying Riesz-Thorin, we obtain the following.

<u>Hausdorff-Young Inequality</u>

If $f \in L^p(\mathbb{R})$, $1 \leq p \leq 2$, then $\hat{f} \in L^{p'}(\mathbb{R})$, $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$\|\hat{f}\|_{p'} \leq \|f\|_p$$

<u>Proof:</u>

We know that $\mathcal{F} : L^1 + L^2 \to L^\infty + L^2$, with

$$\|\mathcal{F}\|_{L^1 \to L^\infty} = 1 \qquad \& \qquad \|\mathcal{F}\|_{L^2 \to L^2} = 1.$$

Fix $1 < p < 2$. Define $\theta = \frac{2p-2}{p}$ & note $0 < \theta < 1$.

Further,

$$\frac{1}{p} = \frac{2-\theta}{2} = \frac{1-\theta}{1} + \frac{\theta}{2}$$

and

$$\frac{1}{p'} = \frac{p-1}{p} = \frac{\theta}{2} = \frac{1-\theta}{\infty} + \frac{\theta}{2}.$$

Hence, Riesz-Thorin implies that $\mathcal{F} : L^p(\mathbb{R}) \to L^{p'}(\mathbb{R})$, with

$$\|\mathcal{F}\| \leq 1^{1-\theta}\, 1^{\theta} = 1. \quad \blacksquare$$

Exercise: $\mathcal{F}: L^p \to L^{p'}$ is the only possibility.

Suppose that $1 \le p, q \le \infty$ & that the F.T. $\mathcal{F}$ is a bounded map of $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$. Show that $q = p'$, i.e., $\frac{1}{p} + \frac{1}{q} = 1$.

Hint: Fix any f & consider $D_r f(x) = r f(rx)$.

Remark

In fact, $\|\mathcal{F}\|_{L^p \to L^{p'}} < 1$ when $1 < p < 2$.

Instead,

$$\|\mathcal{F}\|_{L^p \to L^{p'}} = A_p = \left(\frac{p^{1/p}}{(p')^{1/p'}} \right)^{1/2}.$$

A_p is called the Babenko-Beckner constant (derived by Babenko for some p [p' even] in 1961, all p by Beckner in 1975). The range of $\mathcal{F}: L^p(\mathbb{R}) \to L^{p'}(\mathbb{R})$ is a proper subset of $L^q(\mathbb{R})$.

Remark

If $p > 2$ then $\mathcal{F}$ cannot be defined so that $\mathcal{F}: L^p(\mathbb{R}) \to L^q(\mathbb{R})$ for any q.* Instead, $\mathcal{F}$ maps $L^p(\mathbb{R})$ into the space of tempered distributions. We will consider this in more detail later.

* See Katznelson, p. 123-125 for the analogous question for Fourier series.

__Exercise__ [Katznelson Exercise 3.12, p. 181]
Fix $1 \le p < \infty$. Since $\mathcal{F} : L^p(\mathbb{R}) \to L^{p'}(\mathbb{R})$ is bounded & injective, if it was surjective then $\mathcal{F}^{-1}$ would be continuous by the Inverse Mapping Theorem. Show that $\mathcal{F}^{-1}$ is not bounded, i.e., $\nexists\, C$ s.t.

$$\|f\|_p \le C\,\|\hat{f}\|_{p'}, \quad f \in L^p(\mathbb{R}).$$

Exercise: Gaussians.

Define
$$\varphi_a(x) = e^{-\pi x^2/a}, \qquad a > 0.$$

a. Prove that $\hat{\varphi}_a = a^{1/2} \varphi_{1/a}$.

b. Let $c = a + ib \in \mathbb{C}$. Show that if $\frac{1}{c} = a_0 + ib_0$ then
$$\varphi_c(x) = e^{-\pi x^2/c} = e^{-\pi i b_0 x^2} e^{-\pi a_0 x^2}$$

is a Gaussian φ_{a_0} multiplied by a "chirp" $e^{-\pi i b_0 x^2}$.

c. Prove that $\hat{\varphi}_c = c^{1/2} \varphi_{1/c}$ holds for complex c.

Exercise: $\mathcal{F}$ is not a bounded map of $L^p(\mathbb{R})$ into $L^{p'}(\mathbb{R})$ when $p>2$.

a. For $b>0$, define $f_b(x) = e^{-\pi x^2} e^{-\pi i b x^2}$. Show

$\exists$ constant $C>0$ s.t.
$$\forall b>0, \quad \|\hat{f}_b\|_\infty \le C\, b^{-1/2}.$$

b. Prove for $1<p<2$ that
$$\|\hat{f}_b\|_{p'} \le \|\hat{f}_b\|_2^{2/p'} \, \|\hat{f}_b\|_\infty^{1-\frac{2}{p'}} \le C\, \lambda^{\frac{1}{p'}-\frac{1}{2}}.$$

c. Show $\mathcal{F}$ cannot be a bounded map $L^{p'}(\mathbb{R}) \to L^p(\mathbb{R})$ when $p<2$.

2.4 The Classical Uncertainty Principle

Position: $Pf(x) = x f(x)$

Momentum: $Mf(x) = \frac{1}{2\pi i} f'(x)$ $\qquad (Mf)^\wedge(\xi) = \xi \cdot \hat{f}(\xi)$

Concentration

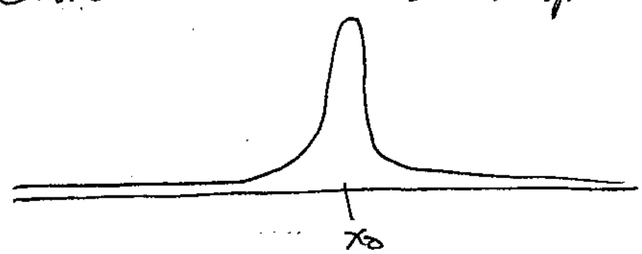
$$\| (x - x_0) f(x) \|_2^2 = \int (x-x_0)^2 \, |f(x)|^2 \, dx$$

measures concentration of f around the point x_0

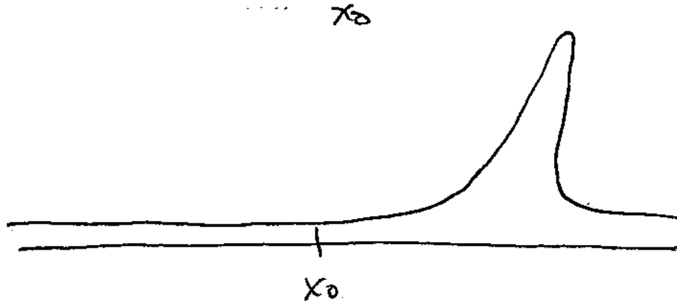
$$\| (\xi - \xi_0) \hat{f}(\xi) \|_2^2 = \int (\xi - \xi_0)^2 \, |\hat{f}(\xi)|^2 \, d\xi$$

measures concentration of $\hat{f}$ around the point ξ_0.

The smaller these quantities are, the more f or $\hat{f}$ is concentrated around the respective point.



$\| (x - x_0) f(x) \|_2^2$ will be small

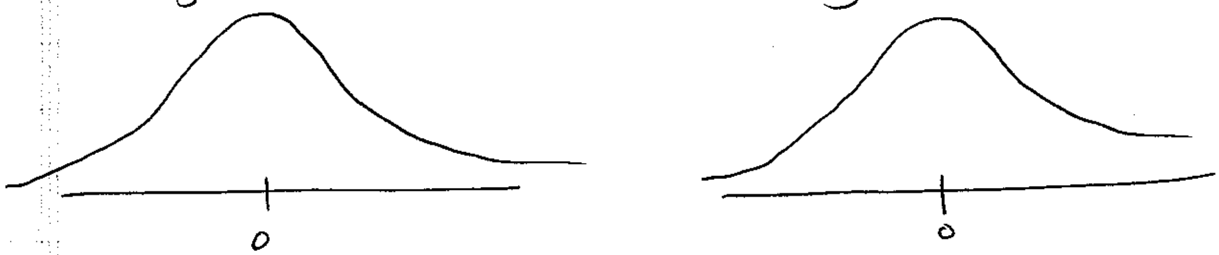


$\| (x - x_0) f(x) \|_2^2$ will be large

Q. How much simultaneous concentration of f & $\hat{f}$ can we have?

Can have quite a lot, e.g., for the Gauss kernel,

$g(x) = e^{-\pi x^2}$ $\qquad\qquad$ $\hat{g}(\xi) = e^{-\pi \xi^2}$



Both are well-concentrated around 0, although they are still "fuzzy peaks".

Can we do better? Can we find f so that

A. NO. The more concentrated f is, the less concentrated $\hat{f}$ must be & vice versa.

Exercise: Think about this in terms of the inversion formula

$$f(x) = \int \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi.$$

Thus, the "fewer" significant values in $f(x)$, the "more" frequencies that must be superimposed to create f.

Classical Uncertainty Principle

If $f \in L^2(\mathbb{R})$ & $x_0, \xi_0 \in \mathbb{R}$, then

$$\text{(UP)} \quad \|f\|_2^2 \leq 4\pi \, \|(x-x_0) f(x)\|_2 \, \|(\xi-\xi_0)\hat{f}(\xi)\|_2$$

In particular,

$$\|f\|_2^2 \leq 4\pi \, \|x f(x)\|_2 \, \|\xi \hat{f}(\xi)\|_2 = 4\pi \|Pf\|_2 \|Mf\|_2$$

Remark: The RHS of (UP) could be infinite.

Exercise: Prove that the RHS of (UP) is ∞ if $f = \chi_{[-T,T]}$.

Proof (~~Wiener~~ Weyl*) $f \in \mathcal{S}(\mathbb{R})$, so

Suppose first that f, $x f(x)$, $f' \in L^1 \cap L^2$,

* Dym & McKean credit this proof to Weyl [1931]

$$\| x f(x) f'(x) \|_1 \leq \| x f(x) \|_2 \; \| f' \|_2 \quad \text{Cauchy-Schwarz}$$

(equality if $x f(x) = r f'(x)$ for some $r \in \mathbb{C}$)

$$= \| x f(x) \|_2 \; \| \widehat{f'} \|_2 \quad \text{Plancherel}$$

$$= \| x f(x) \|_2 \; \| 2\pi i \xi \hat{f}(\xi) \|_2$$

$$= 2\pi \; \| x f(x) \|_2 \; \| \xi \hat{f}(\xi) \|_2 .$$

Now, f is differentiable, so $\overline{f}$ is also, & hence $|f|^2 = f \overline{f}$ is differentiable. Further,

$$-x \, (|f|^2)'(x) = -x \, (f \overline{f})'(x)$$

$$= -x \left(f(x) \overline{f'(x)} + f'(x) \overline{f(x)} \right)$$

$$= 2 \operatorname{Re} \left(-x f(x) \overline{f'(x)} \right)$$

$$\leq 2 \, | x f(x) f'(x) |$$

(equality if $x f(x) \overline{f'(x)} \leq 0$)

Hence

$$4\pi \, \|x f(x)\|_2 \, \|\xi \hat{f}(\xi)\|_2$$

$$\geq \frac{4\pi}{2\pi} \, \| x f(x) \, f'(x) \|_1$$

$$= \int 2 \, |x f(x) f'(x)| \, dx$$

$$\geq \int |x| \, (|f|^2)'(x) \, dx$$

$$\geq -\int x \, (|f|^2)'(x) \, dx$$

$$= -x \, |f(x)|^2 \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} |f(x)|^2 \, dx$$

$$= \|f\|_2^2 \, .$$

Thus, the theorem is proved for $f \in \mathcal{S}(\mathbb{R})$.

Also, if $\|x f(x)\|_2 = \infty$ or $\|\xi \hat{f}(\xi)\|_2 = \infty$ then the result is trivial.

So, suppose $f \in L^2(\mathbb{R})$ & $\|x f(x)\|_2, \|\xi \hat{f}(\xi)\|_2 < \infty$.

Exercise: Show $\exists f_n \in \mathcal{S}(\mathbb{R})$ s.t.

$$f_n \longrightarrow f, \quad x f_n(x) \longrightarrow x f(x), \quad \xi \hat{f}_n(\xi) \longrightarrow \xi \hat{f}(\xi)$$

simultaneously in L^2- norm. [See hints on next page.]

Then use this to show that

$$\| f \|_2^2 \leq 4\pi \, \| x f(x) \|_2 \, \| \xi \hat{f}(\xi) \|_2 .$$

Exercise: This proves the result for the case $x_0 = \xi_0 = 0$. Now prove it for arbitrary values of x_0, ξ_0. ∎

Exercise

Let $Pf(x) = xf(x)$.

a. Show $P(f*g) = Pf*g + f*Pg$.

b. Show that if $k \in \mathcal{S}(\mathbb{R})$, $\int k = 1$, & $k_\lambda(x) = \lambda k(\lambda x)$ then:

i. $\|Pk_\lambda\|_1 \to 0$ as $\lambda \to \infty$
(think about the case where k has compact support first)

ii. If $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, then $f * Pk_\lambda \to 0$ as $\lambda \to \infty$

iii. If $f \in L^p(\mathbb{R})$ & $xf(x) \in L^p(\mathbb{R})$ then $P(f*k_\lambda) \to Pf$ as $\lambda \to \infty$.

The hypothesis $k \in \mathcal{S}(\mathbb{R})$ can be relaxed, what is needed on k?

Exercise

Fix $k \in C_c^\infty(\mathbb{R})$ with $\int k = 1$, & set $k_\lambda(x) = \lambda k(\lambda x)$.

Fix $\theta \in C_c^\infty(\mathbb{R})$ with $\theta(0) = 1$, & set $\theta_\lambda(x) = \theta(x/\lambda)$.

a. Show $\{\hat{\theta}_\lambda\}_{\lambda>0}$ is an approximate identity.

Also note that $k_\lambda, \hat{k}_\lambda, \theta_\lambda, \hat{\theta}_\lambda \in \mathcal{S}(\mathbb{R})$.

b. Prove that if $f \in L^2(\mathbb{R})$ then

$$(f\theta_\lambda) * k_\lambda \longrightarrow f \text{ in } L^2(\mathbb{R}).$$

c. Prove that

$$P\left[(f\theta_\lambda) * k_\lambda\right] \longrightarrow Pf \text{ in } L^2(\mathbb{R}).$$

d. Prove that

$$P\left[\left((f\theta_\lambda) * k_\lambda\right)^\wedge\right] \longrightarrow P\hat{f} \text{ in } L^2(\mathbb{R}).$$

Remark

The standard deviations

$$\Delta_f x = \min_{a \in \mathbb{R}} \left(\int (x-a)^2 |f(x)|^2 \, dx \right)^{1/2}$$

$$\Delta_f \xi = \min_{b \in \mathbb{R}} \left(\int (\xi - b)^2 \, |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2}$$

measure the "concentration" of f & $\hat{f}$.

Exercise:

Show that the minimums occur when

$$a = \bar{x} = \frac{\int x \, |f(x)|^2 \, dx}{\|f\|_2^2}$$

$$b = \bar{\xi} = \frac{\int \xi \, |\hat{f}(\xi)|^2 \, d\xi}{\|f\|_2^2}$$

Hint: Minimum of $F(a)$ occurs when $F'(a) = 0$.

Remark:

$\bar{x}$ is the "expected value" of f &
$\Delta_f x$ is the size of its "essential support"
& similarly for $\bar{\xi}$, $\Delta_f \xi$ w.r.t. $\hat{f}$.

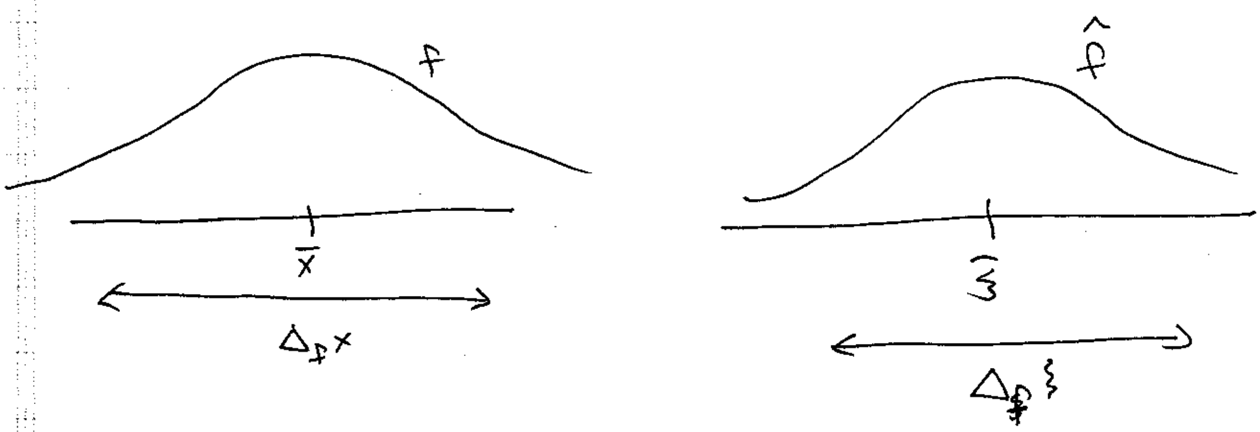
In this notation, the Uncertainty Principle is

$$\|f\|_2 = 1 \implies \Delta_f x \cdot \Delta_{\hat{f}} \xi \geq \frac{1}{4\pi}.$$

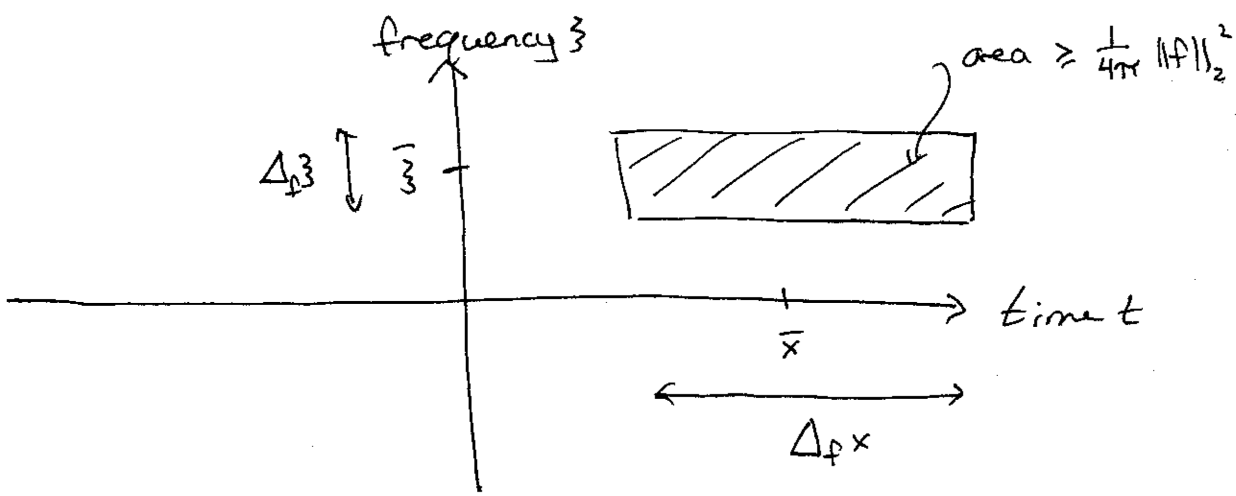
Qualitatively (see Gröchenig, p.29)

"A realizable signal occupies a region of area at least one in the time-frequency plane."

[$\frac{1}{4\pi}$ is due to our normalization choice for the F.T.]



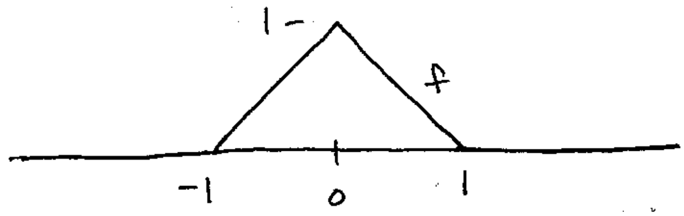
Idealized time-frequency representation of f,



$\mathbb{R}^2$ = t-f plane

Exercise

Compute $\Delta_f x \cdot \Delta_{\hat{f}} \xi$ for the "tent function"



Remark: f is not normalized; divide by $\|f\|_2^2$ first.

Hint: Although f is not differentiable, the a.e. derivative is

$$f' = \chi_{(-1,0)} - \chi_{(0,1)} . \quad \text{a.e.}$$

The hypotheses of the differentiation formula for $\widehat{f'}$ that we have proved do not apply. But you can still directly compute

$$\left(\chi_{(-1,0)} - \chi_{(0,1)}\right)^{\wedge}$$

Exercise: Additive Form of the U.P.

Use the U.P. to show that if $f \in L^2(\mathbb{R})$ then

$$\|Pf\|_2^2 + \|Mf\|_2^2 \geq \frac{1}{2\pi} \|f\|_2^2, \quad (*)$$

with equality if & only if $f(x) = ce^{-\pi x^2}$.

Hint: $2ab \leq a^2 + b^2$.

Exercise:

Prove that if $(*)$ holds $\forall f \in L^2(\mathbb{R})$, then the Classical U.P. follows.

Hint: Replace f with $D_r f(x) = r^{1/2} f(rx)$,

& minimize over $r > 0$.

Thus, the additive form of the U.P. is equivalent to the usual form of the U.P.

Q. Can we achieve equality in (UP)?

A. Case $x_0 = \xi_0 = 0$:

- At the Schwarz step this would require

$$f'(x) = r\,x\,f(x), \qquad r \in \mathbb{C}.$$

Exercise: Solve this DE, show that

$$f(x) = C\,e^{r x^2/2}.$$

Conclude that $\mathrm{Re}(r) < 0$ since $f \in L^2(\mathbb{R})$.

Then we also require, in order to get equality at the 2nd inequality, that $f\bar{f}' \geq 0$.

Exercise: Show this implies $r < 0$.

Exercise: Show $g(x) = C\,e^{-x^2/2}$ yields equality throughout.

Thus the Gaussian $f(x) = C\,e^{-rx^2/2}$ minimizes uncertainty:

$$\|f\|_2^2 = 4\pi\,\|x f(x)\|_2\,\|\xi \hat{f}(\xi)\|_2.$$

Exercise: What if x_0, ξ_0 are arbitrary?

Exercise: Operator-Theoretic Proof of the U.P.

a. Let A, B be self-adjoint but possibly unbounded operators on a Hilbert space H. Prove that

$$\forall a,b \in \mathbb{R}, \quad \forall f \in \text{domain}(AB) \cap \text{domain}(BA),$$

$$\|(A-a)f\| \, \|(B-b)f\| \geq \tfrac{1}{2} \langle [A,B]f, f \rangle, \qquad (*)$$

where $[A,B] = AB - BA$ is the commutator of A & B.

Hint: Show $\langle [A,B]f, f\rangle = 2i \,\mathrm{Im} \langle (B-b)f, (A-a)f\rangle$.

Try $a=b=0$ first. What is $[A-aI, B-bI]$?

Then apply Cauchy-Schwarz.

b. Show that equality holds in $(*)$ if & only if

$(A-a)f = ic(B-b)f$ for some $c \in \mathbb{R}$.

c. Apply part a to the ~~operators~~ position & momentum

operators $Pf(x) = x f(x)$, $Mf(x) = \frac{1}{2\pi i} f'(x)$.

Derive the Uncertainty Principle from this.

See Dym & McKean,
Exercise 10, p. 100

<u>Exercise</u>: Hermite function proof of the U.P.

Recall the Hermite functions:

$$H_n(x) = e^{\pi x^2} D_n e^{-\pi x^2}, \qquad n \geq 0.$$

Let $h_n(x) = H_n(x)/\|H_n\|_2$. Then $\{h_n\}_{n=0}^{\infty}$ is an ONB for $L^2(\mathbb{R})$ & $\hat{h}_n = (-i)^n h_n$. Further, if $Kf(x) = -f''(x) + 4\pi x^2 f(x)$ is the Hermite operator, then $Kh_n = +2\pi(n+1)h_n$.

a. Show that if $f \in \mathcal{S}(\mathbb{R})$, then

$$\langle Kf, f \rangle \geq 2\pi \|f\|_2^2.$$

Hint: Expand the f on the RHS of the inner product in the ONB $\{h_n\}_{n=0}^{\infty}$.

b. Show that if $f \in \mathcal{S}(\mathbb{R})$ then

$$\langle Kf, f \rangle = 4\pi^2 \left(\|x f(x)\|_2^2 + \|\xi \hat{f}(\xi)\|_2^2 \right)$$

Hint: Plancherel/Parseval.

c. Use an extension by density argument to show that the additive form of the U.P. holds:

(*) $$\forall f \in L^2(\mathbb{R}), \qquad \|x f(x)\|_2^2 + \|\xi \hat{f}(\xi)\|_2^2 \geq \frac{1}{2\pi} \|f\|_2^2$$

By an earlier exercise, this implies the U.P.

d. Show that if equality holds in (*) then

$f(x) = c\, e^{-\pi x^2}$.

Hint: How can equality hold in your argument in part a?

Exercise: *Hardy's Inequalities*

a. Given $1 \le p < \infty$ & $\alpha \neq -1$, show $\exists$ constant $C(\alpha,p)$ s.t.

i. If $\alpha < -1$: for any $f \geq 0$,

$$\int_0^\infty \left(\int_0^x f(t)\,dt \right)^p x^\alpha \, dx \le C(\alpha,p) \int_0^\infty f(t)\, t^{\alpha+p}\, dt$$

ii. If $\alpha > -1$: for any $f \geq 0$,

$$\int_0^\infty \left(\int_x^\infty f(t)\,dt \right)^p x^\alpha \, dx \le C(\alpha,p) \int_0^\infty f(t)\, t^{\alpha+p}\, dt.$$

Hints for i: Fix $p+\alpha < \eta < p-1$. Use Hölder to show

$$\left(\int_0^x f(t)\,dt \right)^p \le C \int_0^x f(t)^p\, t^\eta \, dt \; x^{p-1-\eta}.$$

[Alternative approach:

b. For the case $\alpha = -p$, where $1 < p < \infty$, show

$$C(-p,p) \le \left(\frac{p}{p-1} \right)^p = (p')^p$$

Hint: Minimize over η.

c. Define

$$F(x) = \frac{1}{x} \int_0^x f(t)\,dt.$$

Show

$$\|F\|_p \le p' \|f\|_p. \qquad (*)$$

d. Show p' is the best constant in $(*)$.
Hint: Consider $f(t) = t^{-1/p} \chi_{[1,R)}(t)$.

e. Show that equality holds in $(*)$ if & only if $f = 0$ a.e.

Hint: When does equality hold in Hölder's inequality?

Exercise: An L^p uncertainty principle. Fix $1 < p \leq 2$.

a. Prove that if $f \in \mathcal{S}(\mathbb{R})$ & $f(0) = 0$ then

$$\|f\|_2^2 \leq 2\pi p A_p \, \|x f(x)\|_2 \, \|\xi \hat{f}(\xi)\|_2 .$$

Hints: Apply Hölder to $\int_{-\infty}^{\infty} |x f(x)| \, \left|\frac{f(x)}{x}\right| dx$.

Then apply Hardy to $\int_0^{\infty} \left|\frac{f(x)}{x}\right|^{p'} dx$ by

writing $f(x) = \int_0^x f'(t)\, dt$. Lastly, use

Hausdorff-Young.

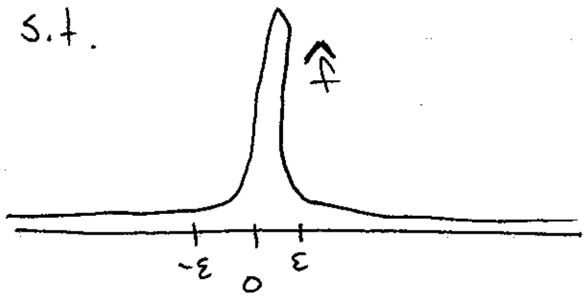
b. Relax the hypotheses on f – what assumptions

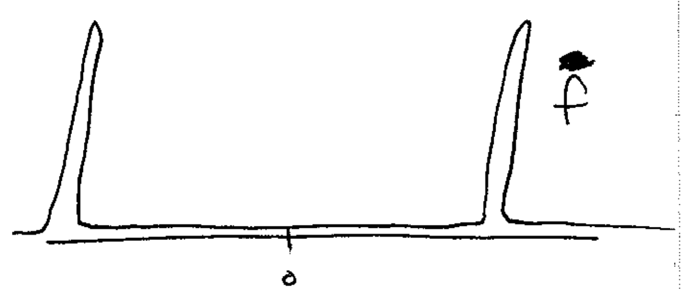
on f are required?

A Local Uncertainty Principle

There are many, many variations on the uncertainty principle. We will give just one in this section.

Q. Could we have an f, say with $\|f\|_2 = \|\hat{f}\|_2 = 1$ s.t.





We would have $\|\xi \hat{f}(\xi)\|_2$ small and $\|x f(x)\|_2$ large, so the product could be large and not violate the U.P.

The following "local U.P." shows this is impossible. In the picture above, $\int_{-\varepsilon}^{\varepsilon} |\hat{f}(\xi)|^2 \, d\xi$ is close to $\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 \, d\xi = 1$, while $4\pi\varepsilon \|f\|_2 \|x f(x)\|_2$ would be "small".

Theorem (Faris, 1978)

If $f \in L^2(\mathbb{R})$ & $\varepsilon > 0$, then for any $\xi_0 \in \mathbb{R}$,

$$\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} |\hat{f}(\xi)|^2 \, d\xi \leq 4\pi\varepsilon \, \|f\|_2 \, \|xf(x)\|_2 .$$

Proof:
This is a simple proof, due to H. Landau, but it does not give the optimal constant. ($8\pi\varepsilon$ instead of $4\pi\varepsilon$).

Assume $f \in \mathcal{S}(\mathbb{R})$. As before,

$$(|\hat{f}|^2)' \leq 2\,|\hat{f}\hat{f}'|.$$

Hence

$$|\hat{f}(\xi)|^2 = \int_{-\infty}^{\xi} (|\hat{f}|^2)'(\eta)\, d\eta \qquad \text{F.T.C.}$$

(f vanishes at $\pm\infty$ since $f \in \mathcal{S}$)

$$\leq \int_{-\infty}^{\xi} 2\,|\hat{f}(\eta)\,\hat{f}'(\eta)|\, d\eta$$

$$\leq 2\,\|\hat{f}\|_2 \, \|\hat{f}'\|_2 \qquad \text{Cauchy-Schwarz}$$

$$= 2\,\|f\|_2 \, \|2\pi i x f(x)\|_2$$

since $(2\pi i x f(x))^\wedge = \hat{f}'$

$$= 4\pi \, \|f\|_2 \, \|xf(x)\|_2 .$$

So,
$$\int_{\xi_0-\varepsilon}^{\xi_0+\varepsilon} |\hat{f}(\xi)|^2 d\xi \le 2\varepsilon \left(4\pi \, \|f\|_2 \, \|x f(x)\|_2 \right)$$
$$= 8\pi\varepsilon \, \|f\|_2 \, \|x f(x)\|_2 .$$

Exercise: Extend by density to all $f \in L^2(\mathbb{R})$. $\blacksquare$

2.5 The Paley-Wiener Theorem

The Paley-Wiener Theorem says that there is a duality under the F.T. between "extreme decay" (compact support) and "extreme smoothness" (extension to an analytic function).

As a corollary, we will obtain the following, which is a type of qualitative uncertainty principle – if f is compactly supported then $\hat{f}$ cannot be, & conversely.

<u>Corollary</u>
If $f \in L^2(\mathbb{R})$, then

$$f, \hat{f} \text{ both have compact support} \iff f = 0 \text{ a.e.}$$

Note that this is not a consequence of the Classical Uncertainty Principle.

<u>Motivation:</u> Fourier series with finitely many terms

Suppose that $f = \{f(n)\}_{n\in\mathbb{Z}}$ is a sequence in $\ell^2(\mathbb{Z})$.

Suppose further that only $f(1), \ldots, f(N)$ are nonzero

(f is a compactly supported <u>sequence</u>).