4. Measures

4.1 Definitions & Properties

A measure assigns a "size" to each element of a collection of subsets of $\mathbb{R}$. Complete definitions & discussion are given in the Appendix; we provide here a brief review of terminology & properties.

Our focus will be on <u>Borel measures</u>, which are defined on the <u>Borel σ-algebra</u> $\mathcal{B}$ of subsets of $\mathbb{R}$. This is the smallest collection of subsets of $\mathbb{R}$ that:

i. contains all the open sets
ii. Is closed under ~~con~~ countable unions, &
iii. Is closed under complements

A function $\mu : \mathcal{B} \longrightarrow [-\infty, \infty]$ is a <u>signed Borel measure</u> if

a. $\mu(\emptyset) = 0$
b. μ takes at most one of the values $\pm\infty$, &
c. If $E_1, E_2, \ldots$ are disjoint elements of $\mathcal{B}$ then

$$\mu\left(\bigcup_k E_k\right) = \sum_k \mu(E_k).$$

If $\mu(E) \geq 0$ for every $E \in \mathcal{B}$, then μ is a <u>positive measure</u> (often simply called a <u>measure</u> in many texts). We write $\mu \geq 0$ in this case.

A function $\mu: \mathcal{B} \to \mathbb{C}$ is a complex Borel measure if

a. $\mu(\emptyset) = 0$

b. If $E_1, E_2, \dots$ are disjoint elements of $\mathcal{B}$ then

$$\mu\left(\bigcup_k E_k\right) = \sum_k \mu(E_k).$$

A signed measure μ is bounded if $|\mu(E)| < \infty \quad \forall E \in \mathcal{B}$. This terminology is due to the fact that if μ is bounded, then

$$(*) \qquad \sup_{E \in \mathcal{B}} |\mu(E)| < \infty.$$

All complex measures are automatically bounded, & they also satisfy $(*)$.

A signed measure μ has a unique Jordan decomposition of the form $\mu = \mu^+ - \mu^-$ where $\mu^+, \mu^- \geq 0$ and $\mu^+ \perp \mu^-$, i.e., $\exists$ disjoint sets A, B s.t. $\mathbb{R} = A \cup B$, $\mu^+(B) = 0$, & $\mu^-(A) = 0$.

The total variation of a signed measure is the positive measure $|\mu| = \mu^+ + \mu^-$. We have

$$|\mu(E)| \leq |\mu|(E) \leq |\mu|(\mathbb{R}) = \|\mu\|, \qquad \forall E \in \mathcal{B};$$

if μ is bounded then $|\mu|(\mathbb{R}) < \infty$ and $(*)$ follows.

The total variation $|\mu|$ of a complex measure is a little less straightforward to define, but again it is a positive measure, & $|\mu(E)| \leq |\mu|(E) \leq |\mu|(\mathbb{R}) = \|\mu\| < \infty$ for all $E \in \mathcal{B}$. If $\mu \geq 0$ then $|\mu| = \mu$.

Integration of real or complex-valued Borel-measurable functions f can be defined w.r.t. a signed or complex Borel measure, & we have that

$$\left|\int f \, d\mu\right| \leq \int |f| \, d|\mu|.$$

$L^1(\mu) = L^1(|\mu|)$ consists of all Borel measurable functions f s.t. $\int |f| \, d|\mu| < \infty$.

A Borel measure μ is <u>locally finite</u> if $|\mu(K)| < \infty$ for each compact set $K \subseteq \mathbb{R}$. All locally finite positive Borel measures are both <u>outer regular</u> & <u>inner regular</u>, i.e., for every $E \in \mathcal{B}$ we have both

$$\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ open}\}$$

and

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}.$$

In the Appendix, the definition of a <u>Radon measure</u> on $\mathbb{R}$ is made. Because the real line is σ-finite & has other properties, TFAE:

a. μ is a locally finite positive Borel measure on $\mathbb{R}$
b. μ is a Radon measure on $\mathbb{R}$.

Also, TFAE:

a.' μ is a bounded positive Borel measure on $\mathbb{R}$.
b.' μ is a bounded Radon measure on $\mathbb{R}$.

These characterizations extend to signed & complex measures. We say that a signed measure ν is a <u>signed Radon measure</u> if the positive measures ν^+, ν^- in the Jordan decomposition $\nu = \nu^+ - \nu^-$ are Radon measures, & we say that a complex measure $\nu = \nu_r + i\nu_i$ is a complex Radon measure if its real & imaginary parts ν_r, ν_i are signed Radon measures.

With this notation, the bounded signed Borel measures on $\mathbb{R}$ coincide with the bounded signed Radon measures, &

the bounded complex Borel measures on $\mathbb{R}$ coincide with the bounded complex Radon measures.

We define

$$M(\mathbb{R}) = \{\nu : \nu \text{ is a complex Radon measure on } \mathbb{R}\}.$$

This is a complete Banach space under the norm

$$\|\nu\| = |\nu|(\mathbb{R}).$$

We have $L^1(\mathbb{R}) \subseteq M^1(\mathbb{R})$ in the sense that each $f \in L^1(\mathbb{R})$ determines a unique bounded Radon measure $\nu_f = f(x)\,dx$. In fact, $L^1(\mathbb{R})$ is identified with the subspace of measures that are absolutely continuous w.r.t. Lebesgue measure, i.e., the range of $f \mapsto f(x)\,dx$ is $\{\nu \in M(\mathbb{R}) : \nu \ll dx\}$.

If we define the complex conjugate of a measure ν by $\bar{\nu}(E) = \overline{\nu(E)}$, then each $\nu \in M(\mathbb{R})$ determines a bounded linear functional on $C_0(\mathbb{R})$ by

$$\langle f, \nu \rangle = \int f(x)\, d\bar{\nu}(x), \qquad f \in C_0(\mathbb{R}).$$

The <u>Riesz Representation Theorem</u> for $C_0(\mathbb{R})$ asserts that all linear functionals on $C_0(\mathbb{R})$ have this form, i.e.,

$$C_0(\mathbb{R})^* = M(\mathbb{R}).$$

Exercise

Show that $M_b(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$.

More precisely, given $\nu \in M_b(\mathbb{R})$, define $L_\nu : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by

$$\langle f, L_\nu \rangle = \int f(x)\, d\bar{\nu}(x), \quad f \in \mathcal{S}(\mathbb{R}).$$

Show that $\nu \longmapsto L_\nu$ is a continuous injection of $M_b(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$.

Exercise

Show that if ν is an arbitrary signed Radon measure (i.e., possibly unbounded), then $L_\nu : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ defined by

$$\langle f, L_\nu \rangle = \int f(x)\, d\nu(x), \quad f \in C_c^\infty(\mathbb{R}),$$

defines a distribution in $\mathcal{D}'(\mathbb{R})$.

Remark

In particular, Lebesgue measure dx is identified with

with the constant function 1 (viewed as a distribution in $\mathcal{D}'(\mathbb{R})$).

Exercise

Show that

$$\delta(E) = \begin{cases} 1, & 0 \in E, \\ 0, & 0 \notin E, \end{cases} \qquad E \in \mathcal{B},$$

defines a bounded positive Borel measure on $\mathbb{R}$.

Exercise

Prove that $M(\mathbb{R})$ is not separable.

Exercise

Show that $\delta' \notin M_b(\mathbb{R})$, i.e., the distribution δ' cannot be extended to be a continuous linear functional on $C_0(\mathbb{R})$.

Exercise

Let $f_k, f \in C_0(\mathbb{R})$ be given. Show that $f_n \xrightarrow{\omega} f$ in $C_0(\mathbb{R})$ if & only if

i. $f_k(x) \longrightarrow f(x) \quad \forall x \in \mathbb{R}$

ii. $\sup_k \|f_k\|_\infty < \infty$.

Hint: Uniform Boundedness Principle.

Exercise

Let $\mu_k, \mu \in M(\mathbb{R})$ be given. Show that $\mu_k \xrightarrow{w^*} \mu$ does not imply $\|\mu_k\| \longrightarrow \|\mu\|$.

Hint: δ_k, point mass at k.

Definition

Given a signed or complex Borel measure μ, we say that μ is concentrated on $E \in \mathcal{B}$ if

$$\forall A \in \mathcal{B}, \quad \mu(A) = \mu(A \cap E).$$

We define

$$\operatorname{supp}(\mu) = \bigcap \{E \in \mathcal{B} : \mu \text{ is concentrated on } E\}$$

Exercise

a. Show that $\operatorname{supp}(\delta) = \{0\}$.

b. Show that if $f \in L^1(\mathbb{R})$, then $\operatorname{supp}(f) = \operatorname{supp}(f\,dx)$, i.e., the support of the function f & the measure $f\,dx$ coincide.

c. Let ν be a signed or complex Radon measure. By an earlier exercise, $\nu \in \mathcal{D}'(\mathbb{R})$. Show that the support of ν as a measure coincides with its support as a distribution.

Exercise

Let S denote the subspace of $M_b(\mathbb{R})$ containing all bounded Radon measures with compact support. Show that S is dense in $M_b(\mathbb{R})$.

Hint: $\|\nu\| = |\nu|(\mathbb{R}) < \infty$, & if $K \subseteq \mathbb{R}$ is compact, then

$$\mu(E) = \nu(E \cap K)$$

defines a measure with compact support.

Remark

Let $\nu \in M_b(\mathbb{R})$ be given. Then the measures $\bar{\nu}$ & $\tilde{\nu}$ are defined by

$$\bar{\nu}(E) = \overline{\nu(E)}, \qquad \tilde{\nu}(E) = \nu(-E), \qquad E \in \mathcal{B}$$

An exercise in the Appendix shows that for ν-integrable f,

$$\int f(x)\, d\bar{\nu}(x) = \overline{\int \overline{f(x)}\, d\nu(x)},$$

& for $\tilde{\nu}$-integrable f (equivalent to $f(-x) \in L^1(\nu)$),

$$\int f(x)\, d\tilde{\nu}(x) = \int f(-x)\, d\nu(x) = \int \tilde{f}(x)\, d\nu(x)$$

Exercise

Let $\mu = \sum'_{n \in \mathbb{Z}} \delta_n$ be the delta train. An earlier exercise showed that $\mu \in \mathcal{S}'(\mathbb{R})$. a. Show that μ is a positive Borel measure on $\mathbb{R}$, & that μ is unbounded $(\mu(\mathbb{R}) = \infty)$.

b. Show directly that $f \mapsto \langle f, \mu \rangle$ is an unbounded linear functional on $C_c(\mathbb{R})$ w.r.t. the L^∞-norm topology on $C_c(\mathbb{R})$.

c. Show directly that $f \mapsto \langle f, \mu \rangle$ is a continuous linear functional w.r.t. the inductive limit topology corresponding to $C_c(\mathbb{R}) = \bigcup_{K \text{ compact}} C(K)$. This is the topology whose convergence criterion is:

$$f_k \longrightarrow f \text{ in } C_c(\mathbb{R}) \text{ if } \quad \begin{array}{l} \text{i. } \exists \text{ compact } K \text{ s.t. } \operatorname{supp}(f_k) \subseteq K \ \forall k \\ \text{ii. } \|f - f_k\|_\infty \longrightarrow 0 \text{ as } k \to \infty. \end{array}$$

4.2 Convolution of Measures

Since a bounded Radon measure ν is a tempered distribution, we already have a definition for the convolution of ν with a Schwartz-class $f \in \mathcal{S}(\mathbb{R})$:

$$(f * \nu)(x) = \overline{\langle T_x \tilde{f}, \nu \rangle}$$

$$= \overline{\int \tilde{f}(y-x) \, d\bar{\nu}(y)}$$

$$= \int f(x-y) \, d\nu(y).$$

For $f \in \mathcal{S}(\mathbb{R})$, $\nu \in M_b(\mathbb{R})$, we have $f * \nu \in C^\infty(\mathbb{R})$ with at most polynomial growth at ∞. But in fact $f * \nu$ is bounded since

$$|(f*\nu)(x)| \le \int |f(x-y)| \, d|\nu|(y)$$

$$\le \|f\|_\infty \int d|\nu|$$

$$= \|f\|_\infty \, \|\nu\|. \qquad (*)$$

But note that ν is also a bounded linear functional on $C_0(\mathbb{R})$ since $M_b(\mathbb{R}) = C_0(\mathbb{R})^*$, and equation (*) says that

$$L_\nu : \mathcal{S}(\mathbb{R}) \, (\subseteq C_0(\mathbb{R})) \longrightarrow L^\infty(\mathbb{R})$$
$$f \longmapsto f * \nu$$

is a bounded map on a dense subspace of $C_0(\mathbb{R})$ (under the L^∞-norm). Hence this map must have a continuous extension to all of $C_0(\mathbb{R})$, i.e., we can define $f * \nu$ for $f \in C_0(\mathbb{R})$, $\nu \in M_b(\mathbb{R})$.

Exercise

Show that this extension is given by the following formula:

If $f \in C_0(\mathbb{R})$, $\nu \in M_b(\mathbb{R})$ then

$$(f * \nu)(x) = \int f(x-y) \, d\nu(y).$$

Hint: Just show that with this definition, $f \longmapsto f * \nu$ is a continuous map of $C_0(\mathbb{R})$ into $L^\infty(\mathbb{R})$. The extension of a bounded linear map on a dense subspace is unique, so this is what it must be.

Exercise

Fix $\nu \in M_b(\mathbb{R})$, & $f \in C_0(\mathbb{R})$. Prove the following.

a. $f * \nu \in C_b(\mathbb{R})$

b. If f, ν both have compact support, then $f * \nu \in C_c(\mathbb{R})$.

c. If either f or ν has compact support, then $f * \nu \in C_0(\mathbb{R})$.

Hint: $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, & the measures with compact support are dense in $M_b(\mathbb{R})$.

d. Show that $f * \nu \in C_0(\mathbb{R})$

Exercise

Fix $\nu \in M_b(\mathbb{R})$. Show that

$$f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}) \implies f * \nu \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}),$$

and in this case,

$$\|f * \nu\|_1 \leq \|f\|_1 \, \|\nu\|$$

Convolution of measures

To motivate our next definition, suppose that $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ and $\nu \in M_b(\mathbb{R})$. Then we have $f*\nu \in C_0(\mathbb{R})$, but also

$$\int |(f*\nu)(x)|\,dx \le \int\int |f(x-y)|\,d|\nu|(y)\,dx$$

$$= \int\int |f(x-y)|\,dx\,d|\nu|(y) \leftarrow \text{Tonelli: OK since } |\nu| \ge 0$$

$$= \int \|f\|_1\,d|\nu|(y)$$

$$= \|f\|_1 \|\nu\| < \infty,$$

so $f*\nu \in L^1(\mathbb{R})$. Hence $f*\nu$ is identified with the measure $f*\nu\,dx$ in $M_b(\mathbb{R})$. Also $d\mu = f\,dx$ defines a measure in $M_b(\mathbb{R})$, so we can regard $f*\nu = \mu*\nu \in M_b(\mathbb{R})$ as the convolution of two measures. Further, as the measure $f*\nu\,dx$ we have

$$(f*\nu)(E) = \int_E (f*\nu)(x)\,dx$$

$$= \int_E \int f(x-y)\,d\nu(y)\,dx$$

$$= \int \int_E f(x-y)\, dx\, d\nu(y)$$

$$= \int \int_{E-y} f(x)\, dx\, d\nu(y)$$

$$= \int \mu(E-y)\, d\nu(y).$$

This suggests the following definition.

<u>Definition</u>

If $\mu, \nu \in M_b(\mathbb{R})$, then $\mu * \nu$ is defined by

$$(\mu * \nu)(E) = \int \mu(E-y)\, d\nu(y), \quad E \in \mathcal{B}.$$

<u>Theorem</u>

If $\mu, \nu \in M_b(\mathbb{R})$, then $\mu * \nu \in M_b(\mathbb{R})$.

<u>Proof:</u>

First note that $(\mu*\nu)(E)$ is well-defined, since

$$\int |\mu|(E-y)\, d|\nu|(y) \le \int \|\mu\|\, d|\nu|(y)$$

$$= \|\mu\|\, \|\nu\| < \infty.$$

We clearly have $(\mu * \nu)(\emptyset) = 0$, so suppose that

$E_1, E_2, \ldots$ are disjoint Borel sets & $E = \bigcup_k E_k$.

Then
$$\sum_{k=1}^{N}{}' \mu(E_k - y) \longrightarrow \sum_{k=1}^{\infty}{}' \mu(E_k - y) = \mu(E-y),$$

and
$$\left| \sum_{k=1}^{N}{}' \mu(E_k - y) \right| \leq \sum_{k=1}^{\infty}{}' |\mu|(E_k - y) = |\mu|(E) \leq \|\mu\| < \infty.$$

Since ν is a bounded measure, the constant function $\|\mu\|$ is $|\nu|$-integrable. Hence by the LDCT,

$$\sum_{k=1}^{\infty}{}' (\mu * \nu)(E_k) = \lim_{N\to\infty} \sum_{k=1}^{N}{}' \int \mu(E_k - y)\, d\nu(y)$$
$$= \int \mu(E-y)\, d\nu(y)$$
$$= (\mu * \nu)(E).$$

Thus $\mu * \nu$ is a complex Borel measure. ∎

Exercises

a. Show that if $\mu, \nu \in M_b(\mathbb{R})$, then $\mu * \nu \in M_b(\mathbb{R})$ and

$$\|\mu * \nu\| \leq \|\mu\| \, \|\nu\|.$$

Thus $M_b(\mathbb{R})$ is a Banach algebra under convolution.

b. Show that if $E \in \mathcal{B}$ then

$$(\mu * \nu)(E) = \iint \chi_E(x+y) \, d\mu(x) \, d\nu(y).$$

c. Show that convolution is commutative & associative.

d. Show that δ is a unit w.r.t. convolution, i.e.,

$$\nu * \delta = \delta * \nu = \nu, \qquad \nu \in M_b(\mathbb{R}).$$

e. Identifying $f \in L^1(\mathbb{R})$ with the measure $f\,dx \in M_b(\mathbb{R})$, we have $L^1(\mathbb{R}) \subseteq M_b(\mathbb{R})$. Show that $L^1(\mathbb{R})$ is an ideal in $M_b(\mathbb{R})$, i.e.,

$$f \in L^1(\mathbb{R}), \ \nu \in M_b(\mathbb{R}) \implies f * \nu \in L^1(\mathbb{R}).$$

In particular, show that $f*\nu$ is the function

$$(f*\nu)(x) = \int f(x-y) \, d\nu(y).$$

f. Show that $L^1(\mathbb{R})$ is a closed ideal in $M_b(\mathbb{R})$.

That is, show that if $f_n \in L^1(\mathbb{R})$ and $f_n\,dx \longrightarrow \nu \in M_b(\mathbb{R})$, then $d\nu = f\,dx$ for some $f \in L^1(\mathbb{R})$.

Exercise

Show that if $\mu, \nu \in M_b(\mathbb{R})$, then

$$\mu, \nu \geqslant 0 \implies \mu * \nu \geqslant 0,$$

and that in this case $\|\mu * \nu\| = \|\mu\| \, \|\nu\|$.

Exercise

a. Suppose $\nu, \mu \in M_b(\mathbb{R})$. Then μ, ν are tempered distributions. If either ν or μ is compactly supported then $\mu * \nu$ is defined as a tempered distribution by

$$(*) \quad \langle f, \mu * \nu \rangle = \langle f * \tilde{\mu}, \nu \rangle, \qquad f \in \mathcal{S}(\mathbb{R}).$$

Show that this formula is valid even if we do not assume that one of μ, ν is compactly supported.

Hint: The compactly supported measures are dense in $M_b(\mathbb{R})$, so $\exists$ compactly supported $\mu_k \in M_b(\mathbb{R})$ s.t. $\mu_k \to \mu$ in $M_b(\mathbb{R})$.

b. Show that $(*)$ extends to $f \in C_0(\mathbb{R})$.

A distributional approach allows us to give a formula for $\langle f, \mu * \nu \rangle$.

<u>Theorem</u>

Let $\mu, \nu \in M_b(\mathbb{R})$ be given. Then

$$\forall f \in C_0(\mathbb{R}), \quad \langle f, \mu * \nu \rangle = \int\int f(x+y)\, d\bar{\mu}(x)\, d\bar{\nu}(y).$$

<u>Proof:</u>

Suppose that $f \in C_0(\mathbb{R})$. Then

$$\langle f, \mu * \nu \rangle = \langle f * \tilde{\mu}, \nu \rangle \qquad \text{(earlier exercise).}$$

$$= \int (f * \tilde{\mu})(y)\, d\bar{\nu}(y)$$

$$= \int \int f(y-x)\, d\tilde{\mu}(x)\, d\bar{\nu}(y)$$

$$= \int \overline{\int \overline{f(y+x)}\, d\mu(x)}\, d\bar{\nu}(y)$$

$$= \int \int f(y+x)\, d\bar{\mu}(x)\, d\bar{\nu}(y). \quad \blacksquare$$

<u>Remark</u>: This shows that if $\mu, \nu \in M_b(\mathbb{R})$, then the definition of their convolution as measures coincides with the definition of their convolution as distributions.

Exercise

Extend the preceding theorem to $C_b(\mathbb{R})$, i.e., show that if $\mu, \nu \in M_b(\mathbb{R})$ & $f \in C_b(\mathbb{R})$ then

$$\langle f, \mu * \nu \rangle = \iint f(x+y)\, d\bar{\mu}(x)\, d\bar{\nu}(y).$$

Remark/Exercise

An alternative approach to the preceding ~~result~~ result would be to directly evaluate $\langle f, \mu * \nu \rangle$ based on the definition of the integral of f w.r.t. the measure $\overline{\mu * \nu}$.

In particular, if μ, ν are positive measures in $M_b(\mathbb{R})$ & $E \in \mathcal{B}$, then

$$\int \chi_E(x)\, d(\mu*\nu)(x) = (\mu*\nu)(E)$$

$$= \int \mu(E-y)\, d\nu(y)$$

$$= \int \int \chi_{E-y}(x)\, d\mu(x)\, d\nu(y)$$

$$= \int \int \chi_E(x+y)\, d\mu(x)\, d\nu(y).$$

Extend this to simple functions, nonnegative functions, real-valued functions, & complex-valued functions in turn to show that for Borel measurable f (assume bounded if necessary):

$$\int f(x)\, d(\mu*\nu)(x) = \iint f(x+y)\, d\mu(x)\, d\nu(y).$$

Extend then to signed measures & complex measures in turn to show that

$$\langle f, \mu*\nu \rangle = \int f(x)\, d(\overline{\mu*\nu})(x) = \iint f(x+y)\, d\bar{\mu}(x)\, d\bar{\nu}(y).$$

Show directly that we also have

$$\langle f * \tilde{\mu}, \nu \rangle = \iint f(x+y)\, d\bar{\mu}(x)\, d\bar{\nu}(y).$$

Exercise

Suppose that $\nu_n \in M_b(\mathbb{R})$ satisfy

$$\lim_{\lambda \to \infty} \left[\sup_n |\nu_n|(\mathbb{R} \setminus [-\lambda, \lambda]) \right] = 0.$$

Show that if $\nu_n \xrightarrow{\omega^*} \nu \in M_b(\mathbb{R})$, then $\hat{\nu}_n(\xi) \to \hat{\nu}(\xi)$ uniformly on compact subsets of $\mathbb{R}$.

Hint: First show pointwise convergence. If $\hat{k}_\lambda \in C_0(\mathbb{R})$, consider

$$\langle e^{2\pi i \xi x} \hat{k}_\lambda(\xi), \nu_n \rangle.$$

4.3 Fourier Transform of Measures

If $\nu \in M_b(\mathbb{R})$ then it is a tempered distribution & therefore has a distributional Fourier transform $\hat{\nu} \in \mathcal{S}'(\mathbb{R})$.
But in fact, the F.T. is given by a function.
To motivate this, consider first a measure $d\nu = f\,dx$ where $f \in L^1(\mathbb{R})$. Then

$$\hat{f}(\xi) = \int f(x)\, e^{-2\pi i \xi x}\, dx = \int e^{-2\pi i \xi x}\, d\nu(x).$$

Identifying f with ν, suggests the following definition.

Definition
The F.T. of $\nu \in M_b(\mathbb{R})$ is

$$\hat{\nu}(\xi) = \int e^{-2\pi i \xi x}\, d\nu(x), \qquad \xi \in \mathbb{R}.$$

Note: The F.T. of measures is often called the <u>Fourier-Stieltjes transform</u> (e.g., see Katznelson).

Note that $\hat{\nu}$ is a well-defined bounded function, since

$$|\hat{\nu}(\xi)| \le \int |e^{-2\pi i \xi x}|\, d|\nu|(x) = \|\nu\| < \infty.$$

Exercise

Show that if $\nu \in M_b(\mathbb{R})$ then $\hat{\nu}$ is uniformly continuous.

Thus, the F.T. is a continuous map of $M_b(\mathbb{R})$ into $C_b(\mathbb{R})$. However, since

$$\hat{\delta}(\xi) = \int e^{-2\pi i \xi x} \, d\delta(x) = e^{-2\pi i \xi \cdot 0} = 1, \quad \xi \in \mathbb{R},$$

we see that the Riemann-Lebesgue Lemma does not extend to measures, & thus the range of $\mathcal{F}: M_b(\mathbb{R}) \to C_b(\mathbb{R})$ is not contained in $C_0(\mathbb{R})$.

Definition

$$B(\hat{\mathbb{R}}) = \{ \hat{\nu} : \nu \in M_b(\mathbb{R}) \}$$

Exercise

$B(\mathbb{R})$ is a Banach algebra w.r.t. pointwise multiplication, under the norm

$$\| \hat{\nu} \|_B = \| \nu \|$$

Theorem

If $\mu, \nu \in M_b(\mathbb{R})$, then $(\mu * \nu)^\wedge = \hat{\mu}\hat{\nu}$.

Proof:

An earlier formula shows that if f is bounded & measurable, then

$$\langle f, \mu * \nu \rangle = \iint f(x+y)\, d\bar{\mu}(x)\, d\bar{\nu}(y).$$

Applying this to $\bar{\mu}, \bar{\nu}$ shows that

$$\int f(x)\, d(\mu * \nu)(x) = \iint f(x+y)\, d\mu(x)\, d\nu(y).$$

Hence

$$\begin{aligned}
(\mu * \nu)^\wedge(\xi) &= \int e^{-2\pi i \xi x}\, d(\mu * \nu)(x) \\
&= \int\int e^{-2\pi i \xi (x+y)}\, d\mu(x)\, d\nu(y) \\
&= \int \left(\int e^{-2\pi i \xi x}\, d\mu(x) \right) e^{-2\pi i \xi y}\, d\nu(y) \\
&= \int \hat{\mu}(\xi)\, e^{-2\pi i \xi y}\, d\nu(y) \\
&= \hat{\mu}(\xi)\, \hat{\nu}(\xi). \quad \blacksquare
\end{aligned}$$

In order to show the relation between the F.T. & the distributional F.T., we need the following result.

<u>Theorem</u> (Parseval Formula)

Let $\nu \in M_b(\mathbb{R})$ be given. If $f, \hat{f} \in L^1(\mathbb{R})$, then

$$\int \hat{f}(\xi)\,\overline{\hat{\nu}(\xi)}\,d\xi = \langle \hat{f}, \hat{\nu} \rangle = \langle f, \nu \rangle = \int f(x)\,d\bar{\nu}(x).$$

<u>Proof:</u>

The hypotheses imply that $f, \hat{f} \in C_0(\mathbb{R})$, so

$\langle f, \nu \rangle = \int f(x)\,d\bar{\nu}(x)$ is defined. Further, the

Inversion Formula applies, i.e.,

$$f(x) = (\hat{f})^{\vee}(x) = \int \hat{f}(\xi)\,e^{2\pi i \xi x}\,d\xi, \quad x \in \mathbb{R},$$

equality holding pointwise everywhere. Hence

$$\begin{aligned}
\int f(x)\,d\bar{\nu}(x) &= \int\int \hat{f}(\xi)\,e^{2\pi i \xi x}\,d\xi\,d\bar{\nu}(x) \\
&= \int\int \hat{f}(\xi)\,e^{2\pi i \xi x}\,d\bar{\nu}(x)\,d\xi \\
&= \int \hat{f}(\xi)\,\overline{\int e^{-2\pi i \xi x}\,d\nu(x)}\,d\xi \\
&= \int \hat{f}(\xi)\,\overline{\hat{\nu}(\xi)}\,d\xi.
\end{aligned}$$

Exercise: Use Fubini's Theorem to justify the interchange in the order of integration. ∎

Exercise

Let $\nu \in M_b(\mathbb{R})$ be given.

a. Show that $\hat{\bar{\nu}}(\xi) = \tilde{\hat{\nu}}(\xi) = \overline{\hat{\nu}(-\xi)}$.

b. Show that if $f, \hat{f} \in L^1(\mathbb{R})$, then

$$\int f(x)\, d\nu(x) = \int \hat{f}(\xi)\, \hat{\nu}(-\xi)\, d\xi.$$

Corollary

If $\nu \in M_b(\mathbb{R})$, then the definition of $\hat{\nu}$ given above coincides with the F.T. of ν as a tempered distribution.

Proof:

If $f \in \mathcal{S}(\mathbb{R})$, then $f, \hat{f} \in L^1(\mathbb{R})$, so by the Parseval Formula,

$\langle \hat{f}, \hat{\nu} \rangle = \langle f, \nu \rangle$. Thus $\hat{\nu}$, the F.T. of ν as a measure, acts on $\mathcal{S}(\mathbb{R})$ just as the distributional F.T. of ν does, & hence it is the distributional F.T. ∎

Corollary: Uniqueness Theorem

If $\nu \in M_b(\mathbb{R})$ & $\hat{\nu} = 0$, then $\nu = 0$.

Proof:

If $\hat{\nu} = 0$, then for every $f \in \mathcal{S}(\mathbb{R})$ we have

$\langle f, \nu \rangle = \langle \hat{f}, \hat{\nu} \rangle = 0$. Hence ν is the zero distribution,

& since the embedding of $M_b(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$ is

injective, we conclude ν is the zero measure.

Alternatively, $\mathcal{S}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$ in L^∞-norm,

so the fact that $\langle f, \nu \rangle = 0$ for $f \in \mathcal{S}(\mathbb{R})$ extends

to all $f \in C_0(\mathbb{R})$. Since $M_b(\mathbb{R}) = C_0(\mathbb{R})^*$, this

implies $\nu = 0$. $\blacksquare$

By using approximate identities, we can extend the Parseval formula as follows.

<u>Theorem</u>

Let $\nu \in M_b(\mathbb{R})$ be given. Then for $f \in L^1(\mathbb{R}) \cap C_b(\mathbb{R})$,

$$\langle f, \nu \rangle = \lim_{\lambda \to \infty} \int \hat{f}(\xi)\, \hat{k}_\lambda(\xi)\, \overline{\hat{\nu}(\xi)}\, d\xi,$$

where $\{k_\lambda\}_{\lambda > 0}$ is a fixed approximate identity that satisfies $\hat{k}_\lambda \in L^1(\mathbb{R})$ $\forall \lambda$.

<u>Proof:</u>

By definition of approximate identity, $M = \sup \|k_\lambda\|_1 < \infty$.

Note that $\|f * k_\lambda\|_\infty \leq \|f\|_\infty \|k_\lambda\|_1 \leq \|f\|_\infty M$.

If $f = 0$ then we are done, so we may assume $\|f\|_\infty > 0$. Fix $\varepsilon > 0$. Then since $|\nu|(\mathbb{R}) < \infty$, $\exists$ compact $K \subseteq \mathbb{R}$ s.t.

$$|\nu|(\mathbb{R} \setminus K) < \frac{\varepsilon}{2(\|f\|_\infty + \|f\|_\infty M)}.$$

By an earlier exercise, since $f \in C_b(\mathbb{R})$ we have that $f * k_\lambda \to f$ uniformly on compact sets.

Since $|f(x) - (f * k_\lambda)(x)| \le \|f\|_\infty + \|f\|_\infty M$ and constants are $|\nu|$-integrable, it therefore follows from the LDCT that $\exists\, \lambda_0$ s.t.

$$\lambda > \lambda_0 \implies \int_K |f(x) - (f*k_\lambda)(x)| \, d|\nu|(x) < \frac{\varepsilon}{2}.$$

Therefore for $\lambda > \lambda_0$ we have

$$\left| \int (f(x) - (f*k_\lambda)(x)) \, d\bar{\nu}(x) \right|$$

$$\le \int_K |f(x) - (f*k_\lambda)(x)| \, d|\nu|(x) + \int_{\mathbb{R}\setminus K} \left(\|f\|_\infty + \|f\|_\infty M \right) d|\nu|(x)$$

$$< \frac{\varepsilon}{2} + \left(\|f\|_\infty + \|f\|_\infty M \right) |\nu|(\mathbb{R}\setminus K)$$

$$< \varepsilon.$$

Thus $\langle f * k_\lambda, \nu \rangle \longrightarrow \langle f, \nu \rangle$ as $\lambda \to \infty$.

Since $f \in L^1(\mathbb{R})$ we have $f * k_\lambda \in L^1(\mathbb{R})$. Also $\hat{f} \in C_0(\mathbb{R})$ & $\hat{k}_\lambda \in L^1(\mathbb{R})$, so $\hat{f}\hat{k}_\lambda \in L^1(\mathbb{R})$. Therefore by the Parseval Formula proved earlier,

$$\langle f * k_\lambda, \nu \rangle = \langle (f * k_\lambda)^\wedge, \nu \rangle$$

$$= \langle \hat{f} \hat{k}_\lambda , \hat{\nu} \rangle$$

$$= \int \hat{f}(\xi) \hat{k}_\lambda(\xi) \overline{\hat{\nu}(\xi)} \, d\xi . \qquad \blacksquare$$

<u>Corollary</u>

If $\nu \in M_b(\mathbb{R})$ & $f \in L^1(\mathbb{R}) \cap C_b(\mathbb{R})$, then

$$\langle f, \nu \rangle = \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} \hat{f}(\xi) \left(1 - \frac{|\xi|}{\lambda} \right) \overline{\hat{\nu}(\xi)} \, d\xi .$$

<u>Proof:</u>

Let $\{k_\lambda\}$ be the Fejer kernel & apply the preceding result. $\blacksquare$

The following criterion is often useful for checking that ~~any~~ the F.T. of a given ~~function~~ measure equals a candidate ~~function~~ function.

<u>Notation</u>

$$\mathcal{F}C_c^\infty(\mathbb{R}) = \{\hat{f} : f \in C_c^\infty(\mathbb{R})\}.$$

Note that
$$\mathcal{F}C_c^\infty(\mathbb{R}) = \{f \in \mathcal{S}(\mathbb{R}) : \hat{f} \text{ is compactly supported}\}$$
$$= \{f \in \mathcal{S}(\mathbb{R}) : \hat{f} \in C_c^\infty(\mathbb{R})\}$$

<u>Lemma</u>

Suppose $\varphi \in C(\mathbb{R})$ & $\nu \in M_b(\mathbb{R})$ are s.t.

$$\forall f \in \mathcal{F}C_c^\infty(\mathbb{R}), \quad \langle \hat{f}, \varphi \rangle = \langle f, \nu \rangle.$$

Then $\hat{\nu} = \varphi$.

<u>Proof:</u>

By an earlier theorem, we can apply the Parseval Equality to conclude that

$$\langle \hat{f}, \varphi \rangle = \langle f, \nu \rangle = \langle \hat{f}, \hat{\nu} \rangle, \quad \text{all } f \in \mathcal{F}C_c^\infty(\mathbb{R}).$$

Hence $\langle f, \varphi \rangle = \langle f, \hat{\nu} \rangle$ for all $f \in C_c^\infty(\mathbb{R})$, so φ & $\hat{\nu}$ define the same distribution, & hence are equal a.e. Since both are continuous functions, they must be equal everywhere.

Alternatively we can see the equality directly. Since $\hat{\nu} - \varphi$ is continuous, its real & imaginary parts are both continuous.

Suppose $\mathrm{Re}\left(\hat{\nu}(\xi_0) - \varphi(\xi_0)\right) > 0$ for some $\xi_0 \in \mathbb{R}$.

Then $\exists$ open interval (a,b) containing ξ_0 s.t.

$\mathrm{Re}\left(\hat{\nu}(\xi) - \varphi(\xi)\right) > 0$ for $\xi \in (a,b)$.

Let $f \in \mathcal{S}(\mathbb{R})$ be s.t. $\mathrm{supp}(\hat{f}) = [a,b]$,

$\hat{f}$ is real-valued & $\hat{f} > 0$ on (a,b). Then

since $f \in \mathcal{F}C_c^\infty(\mathbb{R})$, we have

$$
\begin{aligned}
0 &= \mathrm{Re}\left(\langle \hat{f}, \hat{\nu} \rangle - \langle \hat{f}, \varphi \rangle \right) \\
&= \int \hat{f}(\xi)\, \mathrm{Re}\left(\hat{\nu}(\xi) - \varphi(\xi)\right) d\xi \\
&> 0,
\end{aligned}
$$

a contradiction. Hence $\mathrm{Re}(\hat{\nu} - \varphi) = 0$, &

similarly $\mathrm{Im}(\hat{\nu} - \varphi) = 0$, so $\hat{\nu} = \varphi$ everywhere. $\blacksquare$

Which functions are the Fourier transforms of measures? This is a very difficult question, essentially just as difficult as characterizing $A(\mathbb{R})$. explicitly.

One implicit characterization is the following result. Essentially, a function is the F.T. of a measure if it is the F.T. of a bounded linear functional on $C_0(\mathbb{R})$. However, in order to test boundedness we must work on the F.T. side & hence need to restrict to some dense subset of $C_0(\mathbb{R})$ in order that all the quantities involved will be defined.

<u>Lemma</u> (See Katznelson)

Let $\varphi \in C(\mathbb{R})$ be given. Then TFAE.

a. $\varphi = \hat{\nu}$ for some $\nu \in M_b(\mathbb{R})$, ie., $\varphi \in B(\mathbb{R})$.

b. $\exists\, C > 0$ s.t. $\forall f \in \mathcal{F}C_c^\infty(\mathbb{R})$, $|\langle \hat{f}, \varphi \rangle| \le C \|f\|_\infty$.

<u>Proof:</u>

$a \Rightarrow b$. If $\varphi = \hat{\nu}$ & $f \in \mathcal{F}C_c^\infty(\mathbb{R})$, then by the Parseval Equality we have

$$|\langle \hat{f}, \varphi \rangle| = |\langle \hat{f}, \hat{\nu} \rangle| = |\langle f, \nu \rangle| \le \|f\|_\infty \|\nu\|.$$

$b \Rightarrow a$. Suppose that statement b holds. Define

$\mu : \mathcal{F} C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ by $\langle f, \mu \rangle = \langle \hat{f}, \varphi \rangle$, $f \in \mathcal{F} C_c^\infty(\mathbb{R})$.

Considering $\mathcal{F} C_c^\infty(\mathbb{R})$ as a subspace of $C_0(\mathbb{R})$ under the L^∞-norm, statement b implies that μ is a bounded linear functional on $\mathcal{F} C_c^\infty(\mathbb{R})$. Since $\mathcal{F} C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, $\exists$ unique extension of μ to a bounded linear functional on $C_0(\mathbb{R})$. Thus $\mu \in C_0(\mathbb{R})^* = M_b(\mathbb{R})$, so $\exists$ unique $\nu \in M_b(\mathbb{R})$ s.t. $\langle f, \mu \rangle = \langle f, \nu \rangle$ $\forall f \in C_0(\mathbb{R})$. In particular, for $f \in \mathcal{F} C_c^\infty(\mathbb{R})$, $\langle \hat{f}, \varphi \rangle = \langle f, \nu \rangle$.

The preceding lemma therefore implies that $\varphi = \hat{\nu}$. $\blacksquare$

If $\mu \in M_b(\mathbb{R})$ is a positive measure, then for any $f \geq 0$ s.t. $f, \hat{f} \in L^1(\mathbb{R})$, we have

$$\langle \hat{f}, \hat{\mu} \rangle = \langle f, \mu \rangle = \int f(x)\, d\mu(x) \geq 0.$$

The following result gives a converse to this fact, implicitly characterizing ~~the~~ those bounded continuous functions that are the F.T. of a bounded positive measure.

Theorem

Given $\varphi \in C_b(\mathbb{R})$, TFAE.

a. $\varphi = \hat{\mu}$ for some $\mu \in M_b(\mathbb{R})$ with $\mu \geq 0$.

b. $\langle \hat{f}, \varphi \rangle \geq 0 \quad \forall f \in C_c^\infty(\mathbb{R})$ with $f \geq 0$.

Proof:

$a \Rightarrow b$. Suppose $\varphi = \hat{\mu}$ for some positive measure $\mu \in M_b(\mathbb{R})$. If $f \in C_c^\infty(\mathbb{R})$ with $f \geq 0$, then $f, \hat{f} \in \mathcal{S}(\mathbb{R})$, so by the Parseval Equality we have

$$\langle \hat{f}, \varphi \rangle = \langle \hat{f}, \hat{\mu} \rangle = \langle f, \mu \rangle = \int f(x)\, d\mu(x) \geq 0.$$

$b \Rightarrow a$. Assume that statement b holds. Let

$w(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2$ be the Fejer kernel, &

$W(\xi) = \hat{w}(\xi) = \max\{1 - |\xi|, 0\}$ its Fourier transform.

Set $W_\lambda(\xi) = \lambda W(\lambda \xi)$. Since $W(0) = 1$, we have that $\{W_\lambda\}_{\lambda > 0}$ is an approximate identity.

Since $\varphi \in C_b(\mathbb{R})$, a result from Chapter 3 therefore implies that

$$\lim_{\lambda\to\infty} \langle W_\lambda, \varphi \rangle = \overline{\varphi(0)}.$$

Our goal is to show that $\exists\, C>0$ s.t. $\forall\, f \in C_c^\infty(\mathbb{R})$ we have $\langle \hat{f}, \varphi \rangle \leq C\|f\|_\infty$. An earlier lemma then implies that $\varphi = \hat{\mu}$ for some $\mu \in M_b(\mathbb{R})$, so if we then show that $\mu \geq 0$ then the proof will be complete.

To see this, suppose first that $f \in C_c(\mathbb{R})$ is real-valued. Since

$$\check{W}_\lambda(x) = \omega\left(\frac{x}{\lambda}\right) \longrightarrow 1 \quad \text{as } \lambda \to \infty,$$

uniformly on compact sets, if $\varepsilon>0$ then for λ large enough, say $\lambda > \lambda_0$, we will have

$$\left(\|f\|_\infty + \varepsilon\right) \check{W}_\lambda(x) - f(x) \geq 0.$$

The function

$$g_\lambda(x) = \left(\|f\|_\infty + \varepsilon\right) \check{W}_\lambda(x) - f(x)$$

(in fact, $g_\lambda \in C_b^\infty(\mathbb{R})$),

is continuous, nonnegative, & bounded, but it need not be compactly supported. Let θ be the de la Valee Poussin kernel, & set $\theta_n(x) = n\theta(nx)$. Then $\operatorname{supp}(\check{\theta}_n) = [-2n, 2n]$, $\check{\theta}_n = 1$ on $[-n,n]$, & $0 \leq \check{\theta}_n \leq 1$ everywhere.

Since $\check{\theta}_n g_\lambda \geq 0$ & $\check{\theta}_n g_\lambda \in C_c^\infty(\mathbb{R})$, we have by hypothesis that

$$\langle (\check{\theta}_n g_\lambda)^\wedge, \varphi \rangle \geq 0.$$

Since $\theta_n, \check{\theta}_n, g_\lambda \in L^1(\mathbb{R})$ we have $(\check{\theta}_n g_\lambda)^\wedge = \theta_n * \hat{g}_\lambda$.

And since $\{\theta_n\}_{n\in\mathbb{N}}$ is an approximate identity, & $\hat{g}_\lambda \in L^1(\mathbb{R})$, $\theta_n * \hat{g}_\lambda \longrightarrow \hat{g}_\lambda$ in L^1-norm as $n \to \infty$. Since $\varphi \in C_b(\mathbb{R})$, we therefore have for each $\lambda > \lambda_0$ that

$$\begin{aligned}\langle \hat{g}_\lambda, \varphi \rangle &= \lim_{n\to\infty} \langle \theta_n * \hat{g}_\lambda, \varphi \rangle \\ &= \lim_{n\to\infty} \langle (\check{\theta}_n g_\lambda)^\wedge, \varphi \rangle \\ &\geq 0.\end{aligned}$$

Thus, applying the definition of g_λ, we see that

$$(\|f\|_\infty + \varepsilon)\langle W_\lambda, \varphi\rangle - \langle \hat{f}, \varphi\rangle \geq 0, \quad \forall \lambda > \lambda_0.$$

Letting $\lambda \to \infty$,

$$(*) \quad (\|f\|_\infty + \varepsilon)\overline{\varphi(0)} - \langle \hat{f}, \varphi\rangle \geq 0,$$

Since $-f$ is also real-valued, applying the same argument to $-f$ implies that

$$(**) \quad (\|f\|_\infty + \varepsilon)\overline{\varphi(0)} + \langle \hat{f}, \varphi\rangle \geq 0.$$

Adding $(*)$ to $(**)$ we see that $\varphi(0)$ is real-valued, & subtracting $(*)$ from $(**)$ we see that $\langle \hat{f}, \varphi\rangle$ is real-valued. Combining $(*)$ & $(**)$ therefore implies that

$$|\langle \hat{f}, \varphi\rangle| \leq (\|f\|_\infty + \varepsilon)\varphi(0),$$

so letting $\varepsilon \to 0$ gives

$$|\langle \hat{f}, \varphi\rangle| \leq \|f\|_\infty \varphi(0). \quad (***)$$

Now suppose that $f \in C_c^\infty(\mathbb{R})$ is complex-valued.

Then applying (***) to the real & imaginary parts of f, we see that

$$|\langle \hat{f}, \varphi \rangle| \leq 2\varphi(0) \, \|f\|_\infty, \qquad \text{all } f \in C_c^\infty(\mathbb{R}).$$

An earlier lemma therefore implies that $\varphi = \hat{\mu}$ for some $\mu \in M_b(\mathbb{R})$ [Exercise: Note that that lemma required that the estimate $|\langle \hat{f}, \varphi \rangle| \leq C \|f\|_\infty$ hold for $f \in \mathcal{F} C_c^\infty(\mathbb{R})$ while here we have $f \in C_c^\infty(\mathbb{R})$ – justify applying the lemma].

Finally, the fact that $\langle f, \mu \rangle = \langle \hat{f}, \hat{\mu} \rangle = \langle \hat{f}, \varphi \rangle \geq 0$ for all $f \in C_c^\infty(\mathbb{R})$ with $f \geq 0$ implies by an earlier exercise that $\mu \geq 0$. ∎

<u>Exercise</u> (Katznelson)

Suppose that $\nu_k, \nu \in M_b(\mathbb{R})$. Show that weak* convergence of ν_k to ν does not imply that $\hat{\nu}_k$ converges pointwise to $\hat{\nu}$.

Hint: Consider $\delta_n = T_n\delta$, the point mass at n.

<u>Exercise</u> (Katznelson)

With δ_n the point mass at n measure, define

$$\mu_n = \frac{1}{n}(\delta_1 + \cdots + \delta_n).$$

a. Show that $\mu_n \xrightarrow{w*} 0$.

b. Show that $\lim_{n\to\infty} \hat{\mu}_n(\xi)$ exists $\forall \xi \in \mathbb{R}$, but is not identically zero.

Hint: The limit is $\chi_{\mathbb{Z}}(\xi)$. Note that $\chi_{\mathbb{Z}}$ is discontinuous — compare the next exercise.

Although weak* convergence of measures does not imply pointwise convergence of the corresponding F.T.'s, the following exercise shows that if the F.T.'s do converge to a continuous function, then they converge to the "correct" limit.

Exercise

Suppose $\nu_n, \nu \in M_b(\mathbb{R})$ & $\nu_n \xrightarrow{w^*} \nu$.

Suppose $\varphi \in C(\mathbb{R})$ & $\hat{\nu}_n(\xi) \to \varphi(\xi)$ pointwise.

Show that $\hat{\nu} = \varphi$.

Hints: Weak* convergent sequences are bounded (by the Uniform Boundedness Principle). Show that $\langle \hat{f}, \varphi \rangle = \langle f, \nu \rangle \quad \forall f \in \mathcal{F} C_c^\infty(\mathbb{R})$.

Exercise

Let $\nu \in M_b(\mathbb{R})$ be given, & define $\tilde{\nu} \in M_b(\mathbb{R})$ by

$\tilde{\nu}(E) = \overline{\nu(-E)}$, $E \in \mathcal{B}$. Show that

$$\hat{\tilde{\nu}}(\xi) = \overline{\hat{\nu}(\xi)}, \qquad \xi \in \mathbb{R}.$$

Hint: An exercise in the appendix shows that if $f \in L^1(\tilde{\nu})$ then $\int f(x)\, d\tilde{\nu}(x) = \overline{\int \overline{f(-x)}\, d\nu(x)}$.

Exercise (Folland, p. 272)

a. Show that if $\mu \in M_b(\mathbb{R})$ & $\{k_\lambda\}_{\lambda>0}$ is an approximate identity, then $k_\lambda * \mu \xrightarrow{w^*} \mu$.

b. Show that $L^1(\mathbb{R})$ is weak* dense in $M_b(\mathbb{R})$.

Exercise (Folland, p. 272)

Let δ_x be the point mass at x measure $(x \in \mathbb{R})$.

Let S be the set of all finite linear combinations of δ_x.

Show S is weak* dense in $M_b(\mathbb{R})$.

Hints: Start with $g \in C_c(\mathbb{R})$. Show $\exists\, x_j^n, c_j^n$ s.t.
$$\left\langle f, \sum_{j=1}^{n} c_j^n \delta_{x_j^n} \right\rangle \longrightarrow \langle f, g \rangle$$
by approximating with a Riemann sum. Then C_c is dense in L^1, & L^1 is weak* dense in $M_b(\mathbb{R})$ by the preceding exercise.

Exercise

This is an alternate proof that $(\mu * \nu)^\wedge = \hat{\mu}\,\hat{\nu}$.

a. Show that if $f \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ & $\mu \in M_b(\mathbb{R})$ then $(f*\mu)^\wedge = \hat{f}\,\hat{\mu}$.

Note: An earlier exercise shows that $f*\mu \in L^1 \cap C_0$.

b. Show that if $f \in \mathcal{S}(\mathbb{R})$ & $\mu, \nu \in M_b(\mathbb{R})$, then

$$\langle \hat{f}, (\mu*\nu)^\wedge \rangle = \langle \hat{f}, \hat{\mu}\,\hat{\nu} \rangle.$$

Note that $\mu, \nu, \mu*\nu \in \mathcal{S}'(\mathbb{R})$, so the distributional Plancherel Formula applies, as well as the fact that $\langle f, \mu*\nu \rangle = \langle f*\tilde{\mu}, \nu \rangle$.

c. Use this result to give another proof that $\mu*\nu = \nu*\mu$, by appealing to the Uniqueness Theorem

Exercise (Folland, p. 272)

Let $\nu_k, \nu \in M_b(\mathbb{R})$ be given.

a. Show that if $\sup_k \|\nu_k\| < \infty$ & $\hat{\nu}_k(\xi) \to \hat{\nu}(\xi)$ pointwise, then $\nu_k \xrightarrow{w^*} \nu$.

b. Show that if $\nu_k \xrightarrow{w^*} \nu$ & $\|\nu_k\| \to \|\nu\|$, then $\hat{\nu}_k(\xi) \to \hat{\nu}(\xi)$ pointwise.

Hints: By one of the characterizations of $|\nu|$, find a $g \in C_c(\mathbb{R})$ s.t. $\|g\|_\infty \le 1$ and $\left|\int g\, d\nu\right| \geq \|\nu\| - \varepsilon$. [By multiplying g by a scalar, $\int g\, d\nu$ can be assumed to be real.]

Then $\int e^{-2\pi i \xi x} g(x)\, d\nu_k(x) \longrightarrow \int e^{-2\pi i \xi x} g(x)\, d\nu(x)$.