Reference: Gröchenig
Foundations of Time-Frequency Analysis.

6. Discrete Time-Frequency Analysis

We have seen that if we fix a nonzero window $g \in L^2(\mathbb{R})$, then we can <u>analyze</u> functions $f \in L^2(\mathbb{R})$ via the STFT. The mapping $f \mapsto V_g f$ is a multiple of an isometry into $L^2(\mathbb{R}^2)$:

$$\|V_g f\|_2 = \|f\|_2 \, \|g\|_2.$$

Further, we can <u>synthesize</u> f from its STFT: if we choose any $\gamma \in L^2(\mathbb{R})$ with $\langle g, \gamma \rangle \neq 0$, then

$$f = \iint V_g f(x,\xi) \, M_\xi T_x \gamma \; dx \, d\xi, \qquad f \in L^2(\mathbb{R}). \quad (*).$$

These formulas are continuous analogues of the familiar decomposition induced by an ONB $\{e_n\}_{n \in \mathbb{N}}$. In particular, given any such ONB, we have the Plancherel formula

$$\sum_{n=1}^{\infty} |\langle f, e_n \rangle|^2 = \|f\|_2^2, \qquad f \in L^2(\mathbb{R})$$

and the representation

$$f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n, \qquad f \in L^2(\mathbb{R}).$$

Here the analysis mapping is

$$\begin{aligned} C: \ L^2(\mathbb{R}) &\longrightarrow \ell^2 \\ f &\longmapsto \{\langle f, e_n\rangle\}_{n\in\mathbb{N}}, \end{aligned}$$

& synthesis is the adjoint of analysis, $D = C^*$ with

$$\begin{aligned} D: \ \ell^2 &\longrightarrow L^2(\mathbb{R}) \\ \{c_n\}_{n\in\mathbb{N}} &\longmapsto \sum_{n=1}^{\infty} c_n e_n \end{aligned}$$

Comparing these formulas to those for the STFT, we ask whether there are "discrete" versions of the STFT analysis & reconstruction formulas, analogous to those given by an ONB. One intuition for why this might be true is that there is a great deal of overlap in the time-frequency plane of the building blocks $M_\xi T_x \gamma$ in (*). In particular, we know that fixing the Gaussian window $\phi(t) = 2^{1/4} e^{-\pi t^2}$ for convenience), the essential support of $V_\phi \gamma$ occupies a region of area at least 1 in the time-frequency plane. In absolute value, $V_\phi(M_\xi T_x \gamma)$ is just a translation of $V_\phi \gamma$, so there is a huge overlap in the essential supports of the building blocks $M_\xi T_x \gamma$ in the time-frequency plane. Can we reduce from the uncountable collection of all building blocks $\{M_\xi T_x \gamma\}_{(x,\xi)\in\mathbb{R}^2}$ to a countable set, such as $\{M_{\beta n} T_{\alpha k} \gamma\}_{k,n\in\mathbb{Z}}$?

There are two separate issues here, each expressing a "stability" question in different & complementary ways. In fact, there is also a third stability issue encompassed in the question of whether we can go beyond just the Hilbert space setting, as in our discussion of the modulation spaces in the preceding chapter but for now let us focus on the issues in the Hilbert space setting of $L^2(\mathbb{R})$.

<u>Issue #1</u>: Norm Equivalence

If $\|g\|_2 = 1$, then we have the norm-preservation property that

$$\iint |V_g f(x,\xi)|^2 \, dx \, d\xi = \int |f(t)|^2 \, dt, \quad f \in L^2(\mathbb{R}).$$

We can hope to "discretize" this, to replace the integral by a summation over $V_g f$ evaluated at countably many points say $\{(\alpha k, \beta n)\}_{k,n \in \mathbb{Z}}$. Thus we may hope for

$$\sum_{k\in\mathbb{Z}}{}' \sum_{n\in\mathbb{Z}}{}' |V_g f(\alpha k, \beta n)|^2 = \|f\|_2^2, \quad f \in L^2(\mathbb{R}),$$

or, equivalently,

$$\sum_{k\in\mathbb{Z}}{}' \sum_{n\in\mathbb{Z}}{}' |\langle f, M_{\beta n} T_{\alpha k} g \rangle|^2 = \|f\|_2^2. \qquad (**)$$

We refer to $\{\langle f, M_{\beta n} T_{\alpha k} g \rangle\}_{k,n\in\mathbb{Z}}$ as the <u>Gabor coefficients</u> w.r.t. the <u>Gabor system</u>

$$\mathcal{G}(g,\alpha,\beta) = \{M_{\beta n} T_{\alpha k} g\}_{k,n\in\mathbb{Z}}.$$

Equation (**) would hold if, for example, $\mathcal{G}(g,\alpha,\beta)$ was an ONB for $L^2(\mathbb{R})$. There do exist g, α, β for which $\mathcal{G}(g,\alpha,\beta)$ is an ONB compare the following exercise.

<u>Exercise</u>

$\mathcal{G}(\chi_{[0,1]}, 1, 1)$ is an ONB for $L^2(\mathbb{R})$.

Note:
$$\mathcal{G}(\chi_{[0,1]}, 1, 1) = \{M_n T_k \chi_{[0,1]}\}_{k,n\in\mathbb{Z}}$$
$$= \{e^{2\pi i n x} \chi_{[n,n+1]}(x)\}_{k,n\in\mathbb{Z}}.$$

Since $\{e^{2\pi i n x} \chi_{[n,n+1]}\}_{k\in\mathbb{Z}}$ is an ONB for $L^2[n,n+1]$, we are creating an ONB for $L^2(\mathbb{R})$ by piecing together ONBs for $L^2[n,n+1]$, $n\in\mathbb{Z}$.

Unfortunately, the elements of this ONB are discontinuous and hence are poorly localized in frequency. Alternatively, their STFTs are poorly localized in the time-frequency plane.

This phenomena is typical - all Gabor ONBs are poorly localized in the time-frequency plane (this is addressed by the Balian-Low Theorems discussed in more detail later).

Perhaps we can avoid this phenomenon by relaxing some requirement. An ONB combines two requirements - first, the requirement of orthogonality & second, the basis requirement, corresponding to unique expansions. Perhaps we can ask only that our Gabor system $\mathcal{G}(g, \alpha, \beta)$

preserve, up to some scalar factors, the energy of the Gabor coefficients. That is, perhaps we can ask only that the energy of the Gabor coefficients & the original norm of f be equivalent, i.e.,

$$\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} |\langle f, M_{\beta n}T_{\alpha k}\, g\rangle|^2 \asymp \|f\|_2^2 .$$

This idea leads to the concept of <u>frames</u> of time-frequency shifts.

<u>Issue 2</u>

However, this idea leaves entirely unresolved the issue of whether we have some <u>reconstruction</u> of f from the Gabor coefficients $\{\langle f, M_{\beta n}T_{\alpha k}\, g\rangle\}_{k,n\in\mathbb{Z}}$. Will it be the case that we have

$$f = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle f, M_{\beta n}T_{\alpha k}\, g\rangle\, M_{\beta n}T_{\alpha k}\, \gamma,$$

which is a "discrete" form of the reconstruction formula (*).

This question seems to suggest that we are asking whether $\{M_{\beta n} T_{\alpha k} \gamma\}_{k,n \in \mathbb{Z}}$ will form a <u>Schauder basis</u> for $L^2(\mathbb{R})$, i.e., whether every $f \in L^2(\mathbb{R})$ can be written

$$f = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} c_{kn} M_{\beta n} T_{\alpha k} \gamma$$

w.r.t. some fixed ordering of these series. This again turns out to be too much to ask — Gabor systems $\mathcal{G}(\gamma, \alpha, \beta)$ which are Schauder bases do exist, but they are not well-localized in the time-frequency plane.

Again we ask: what do we really want? A Schauder basis requires an expansion with <u>unique</u> coefficients. Perhaps we can ask only that

$$f = \sum_{k\in\mathbb{Z}}' \sum_{n\in\mathbb{Z}}' \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k} \gamma, \quad f \in L^2(\mathbb{R})$$

hold, without the requirement that the coefficients $\{\langle f, M_{\beta n} T_{\alpha k} g\rangle\}_{k,n\in\mathbb{Z}}$ be unique. Perhaps by relaxing this requirement the series above may even converge regardless of ordering, i.e., unconditionally. Again this turns out to lead us to the concept of Gabor frames.

History
Dennis Gabor (1946, Nobel prize for holography) proposed exactly such an approach, based on using the Gaussian window $\phi(t) = 2^{1/4} e^{-\pi t^2}$, $\alpha=\beta=1$. [Pronunciation: G $\bar{a}$ - bor, long a as in ahh.] The condition $\alpha=\beta=1$ is natural, as it is necessary for the Gabor system to be a basis

(as we will see later). However, it was much later seen that while every $f \in L^2(\mathbb{R})$ can be written

$$f = \sum_{k\in\mathbb{Z}}{}' \sum_{n\in\mathbb{Z}}{}' c_{kn} M_n T_k \phi,$$

this series converges only in the sense of distributions and NOT in L^2-norm. Further the coefficients $\{c_{kn}\}$ grow with k & n, i.e., they are not an L^2-sequence [Janssen.].

Later it was shown that No "nice" window g, γ can be used to give nonredundant Gabor expansions. We must move to redundant Gabor systems that form frames for $L^2(\mathbb{R})$.

In honor of Gabor, time-frequency analysis is often called Gabor analysis.

In summary, there are two separate issues that we need to address, each of which expresses a different kind of stability requirement.

Issue #1.

Is the mapping from f to its sequence of Gabor coefficients $\{\langle f, M_{\beta n} T_{\alpha k}\, g\rangle\}_{k,n\in\mathbb{Z}}$ both continuous & has a continuous inverse, i.e., can we achieve

$$\|f\|_2^2 \asymp \sum_{k\in\mathbb{Z}}' \sum_{n\in\mathbb{Z}}' |\langle f, M_{\beta n} T_{\alpha k}\, g\rangle|^2, \quad f\in L^2(\mathbb{R}),$$

Issue #2

Do we have a basis-like reconstruction of f from its Gabor coefficients, i.e., can we find a suitable window γ s.t.

$$f = \sum_{k\in\mathbb{Z}}' \sum_{n\in\mathbb{Z}}' \langle f, M_{\beta n} T_{\alpha k}\, g\rangle\, M_{\beta n} T_{\alpha k}\, \gamma, \quad f\in L^2(\mathbb{R}),$$

where the series is required to converge in L^2-norm and further must converge unconditionally (regardless of ordering)?

As for an ONB, & because $L^2(\mathbb{R})$ is a <u>Hilbert space</u>, these two issues are closely related. However, to achieve them using "nice" windows (which is required to extend <u>beyond</u> just $L^2(\mathbb{R})$), we must relinquish hopes both of orthogonality and of uniqueness in the basis-like expansions. Instead, we need to construct redundant <u>Gabor frames</u>. These possess all the stability properties of unconditional bases, but without the requirement of unique expansions.

Back to H. Special case: $f_n = \tilde{f}_n$

Given a basis $\{f_n\}_{n\in\mathbb{N}}$ with biorthogonal basis $\{\tilde{f}_n\}_{n\in\mathbb{N}}$.
When $f_n = \tilde{f}_n$, the sequence $\{f_n\}$ is orthonormal, & we have

$$f = \sum_n{}' \langle f, f_n \rangle f_n \qquad \forall f \in H.$$

Hence

$$\|f\|^2 = \langle f, f \rangle = \sum_n{}' \langle f, f_n\rangle \langle f_n, f\rangle = \sum_n{}' |\langle f, f_n\rangle|^2.$$

This is the Plancherel formula for ONB.

Theorem
Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of elements of H. The TFAE:

1. $\{f_n\}$ is an ONB for H.

2. $\{f_n\}$ is ON & $f = \sum' \langle f, f_n \rangle f_n \quad \forall f \in H$

3. $\{f_n\}$ is ON & $\sum' |\langle f, f_n\rangle|^2 = \|f\|^2 \quad \forall f \in H$

4. $\{f_n\}$ is ON & $\nexists\, g \neq 0$ s.t. $g \perp f_n \ \forall n$.

Completeness

Let $\{f_n\}$ be a basis for H with dual basis $\{\tilde{f}_n\}$.

Then
$$f = \sum_{n=1}^{\infty} \langle f, \tilde{f}_n \rangle f_n = \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, \tilde{f}_n \rangle f_n$$

(limit in norm of H).

Thus every $f \in H$ can be approximated arbitrarily closely by a finite linear combination of the f_n, i.e., the set of finite linear combinations is dense in H.

Definition

Let $\{f_n\}_{n=1}^{\infty}$ be any sequence in a Hilbert space H.

a. The finite linear span of $\{f_n\}$ is

$$\mathrm{span}\{f_n\} = \left\{ \sum_{n=1}^{N} c_n f_n \ : \ N>0,\ c_1,\dots,c_N \in \mathbb{C} \right\}$$

b. The closed linear span of $\{f_n\}$ is the closure of the finite linear span, i.e.,

$$\overline{\mathrm{span}}\{f_n\} = \left\{ g \ : \ g = \lim_{N\to\infty} g_N \ \text{ some } g_N \in \mathrm{span}\{f_n\} \right\}.$$

c. $\{f_n\}$ is <u>complete</u> (or <u>total</u> or <u>fundamental</u>) in H if the finite linear span is dense, i.e., if

$$\overline{\text{span}}\,\{f_n\} = H.$$

<u>Example</u>: Every basis is complete.

However, not every complete set is a basis.
For example, H is trivially complete in H.

<u>Conjecture</u>: A complete set that is independent is a basis.

Problem: Independence means ... what?

Theorem

In an ∞-dimensional Hilbert space, if $\{f_n\}$ is complete then:

$\{f_n\}$ is a basis

$$\Downarrow \quad \not\Uparrow$$

$\{f_n\}$ is minimal, i.e., $\forall m,\ f_m \notin \overline{\mathrm{span}}\{f_n\}_{n\neq m}$

$$\Downarrow \quad \not\Uparrow$$

$\{f_n\}$ is ω-independent, i.e.,

if $\sum_{n=1}^{\infty} c_n f_n$ converges & $=0$, then $c_n = 0\ \forall n$

$$\Downarrow \quad \not\Uparrow$$

$\{f_n\}$ is finitely independent, i.e.,

every finite subset of $\{f_n\}$ is linearly independent.

Proof omitted, but the point is there are many shades of gray in the notion of independence in the ∞-dimensional setting that do not appear in the finite-dimensional setting.

<u>Exercise</u>: Given an arbitrary sequence $\{f_n\}$,

$$\{f_n\} \text{ is complete} \iff \nexists\, g \neq 0 \text{ s.t. } \langle g, f_n \rangle = 0 \text{ for every } n.$$

<u>Corollary</u>
If $\{f_n\}$ is an ~~[illegible]~~ <u>orthonormal</u> sequence, then

$$\{f_n\} \text{ is complete} \iff \{f_n\} \text{ is a basis.}$$

That is, with the hypothesis of <u>orthogonality</u>, a complete set is a basis. Without orthogonality this can fail. The problem in the "shades of gray" is that the angle between f_{n+1} & $\text{span}\{f_1, \dots, f_n\}$ can decrease with n — independence is maintained by the possibility of convergent expansions $f = \sum_{n=1}^{\infty} c_n f_n$ is lost.

Final illustration of basis vs. complete set

In a basis, $f = \sum_{n=1}^{\infty} c_n(f) f_n$, i.e., given ε,

$\exists N$ s.t. $\| f - \sum_{n=1}^{N} c_n(f) f_n \| < \varepsilon$.

If we decrease ε, we just take a bigger N to achieve the same inequality. The coeff. $c_n(f)$ are determined by f alone, not by ε. Only the number N of them needed depends on ε.

By contrast, in a complete set all we know is that given ε, there exist coeff. $c_n(f,\varepsilon)$ & an N s.t. $\| f - \sum_{n=1}^{N} c_n(f,\varepsilon) f_n \| < \varepsilon$. If we decrease ε, we may need to both increase N & choose <u>completely different</u> $c_n(f,\varepsilon)$ to achieve the same inequality. The coeff. depend on both f & ε.

6.1 Abstract frames in Hilbert Spaces; Frames vs. Bases

Definition

A set of vectors $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space H is a frame

if $\exists\, A, B > 0$ (called frame bounds) s.t.

$$\forall f \in H, \qquad A\,\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\,\|f\|^2,$$

i.e., $\|f\|$ & the ℓ^2-norm of the coefficients $\{\langle f, f_n\rangle\}_{n=1}^{\infty}$

are "equivalent" (but not necessarily equal).

If $A = B$ then the frame is said to be tight.

If the frame is a basis (defined precisely below), then the frame is said to be exact.

Example: If $\{f_n\}$ is an ONB for H then it is a tight frame with $A = B = 1$.

However, a frame with $A = B = 1$ need not be an ONB!

Example: If $\{f_n\}$, $\{g_n\}$ are both ONB for H, then

$\{f_n\} \cup \{g_n\}$ is a tight frame with $A = B = 2$.

Renormalize to get $A = B = 1$, but its not ON or a basis anymore.

We'll see that if $\{f_n\}$ is a frame then $\exists\, \tilde{f}_n \in H$ s.t.

$$f = \sum_n{}' \langle f, f_n \rangle \tilde{f}_n = \sum_n{}' \langle f, \tilde{f}_n \rangle f_n. \qquad (*)$$

That is, from the norm-equivalence condition in the def. of frame, we automatically get basis-like expansions. However, the expansions in $(*)$ need <u>not</u> be unique, so we need not get true bases.

Before proceeding, let's review basic results on bases.

<u>Definition</u>

A sequence $\{f_n\}_{n=1}^{\infty}$ in a Hilbert space H is a <u>basis</u> for H if

$$\forall f \in H, \ \exists \ \underline{\text{unique}} \ c_n(f) \in \mathbb{C} \quad \text{s.t.} \quad f = \sum_{n=1}^{\infty} c_n(f) f_n.$$

<u>Remarks.</u>

1. Same definition for basis in a Banach space X; certain aspects of what we will discuss carry over to general Banach spaces but often you have to distinguish carefully between X & its dual space X'.

2. The basis def. starts from expansions whereas the frame def. starts from the norm equivalence. Frames lead to basis-like expansions, but bases in general do not lead to norm equivalences. The subclass of <u>Riesz bases</u> (defined later) turn out to be the ones which satisfy a norm-equivalence condition, i.e.

   Riesz basis $\Longleftrightarrow$ frame + basis $\Longleftrightarrow$ exact frame.

3. $f = \sum_{n=1}^{\infty} c_n(f) f_n$ means convergence in the norm of H, i.e.,

$$\lim_{N \to \infty} \left\| f - \sum_{n=1}^{N} c_n(f) f_n \right\| = 0.$$

4. Basis in this context does not mean vector space basis.

A vector space basis, or a Hamel basis, of H is a collection $\{f_\alpha\}$ s.t.

Every finite subset of $\{f_\alpha\}$ is linearly independent.

&

Every $f \in H$ can be written as a finite linear combination of f_α, i.e., $f = \sum_{k=1}^{N} c_k f_{\alpha_k}$.

The Axiom of Choice is equivalent to the stmt that every vector space has a Hamel basis, but if H is ∞-dimensional, then a Hamel basis for H must be uncountable, & no constructive proof of existence exists.

Theorem

Every basis $\{f_n\}$ for H (or for a general Banach space X) is a Schauder basis, i.e., ~~basis~~ for each fixed n, the mapping $n \mapsto c_n(f)$ is a continuous linear functional on H. Hence $c_n \in H' = H$ (for a general Banach space only get $c_n \in X'$ here),

which means $\exists\, \tilde{f}_n \in H$ s.t. $c_n(f) = \langle f, \tilde{f}_n \rangle$.

Thus

$$f = \sum_{n=1}^{\infty} \langle f, \tilde{f}_n \rangle f_n,$$

i.e., the coefficients in the expansion of f are given by inner products against the "dual system" $\tilde{f}_n$.

In fact, $\{\tilde{f}_n\}$ is also a basis for H, &

$$f = \sum_{n=1}^{\infty} \langle f, \tilde{f}_n \rangle f_n = \sum_{n=1}^{\infty} \langle f, f_n \rangle \tilde{f}_n.$$

<u>Biorthogonality</u>

With m fixed, we have

$$f_m = \sum_{n=1}^{\infty} \langle f_m, \tilde{f}_n \rangle f_n \quad \& \quad f_m = \sum_{n=1}^{\infty} \delta_{mn} f_n.$$

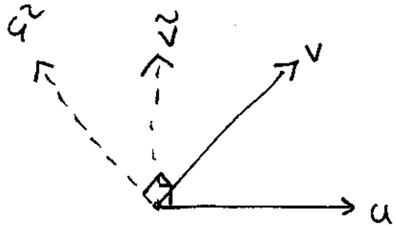
Since $\{f_n\}$ is a basis, there is a <u>unique</u> expansion of f_m in the basis elements, so we conclude

$$\langle f_m, \tilde{f}_n \rangle = \delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

We say that $\{f_n\}$ & $\{\tilde{f}_n\}$ are biorthogonal sequences.
Note how uniqueness is critical in obtaining this result.

Example: Finite-dimensions, say $\mathbb{R}^2$.

Any linearly independent spanning set is a basis.



$\tilde{u} \perp v$ $\qquad$ $\tilde{u} \cdot u = 1$

$\tilde{v} \perp u$ $\qquad$ $\tilde{v} \cdot v = 1$

If $x \in \mathbb{R}^2$ then $X = au + bv$ for some $a, b \in \mathbb{R}$.

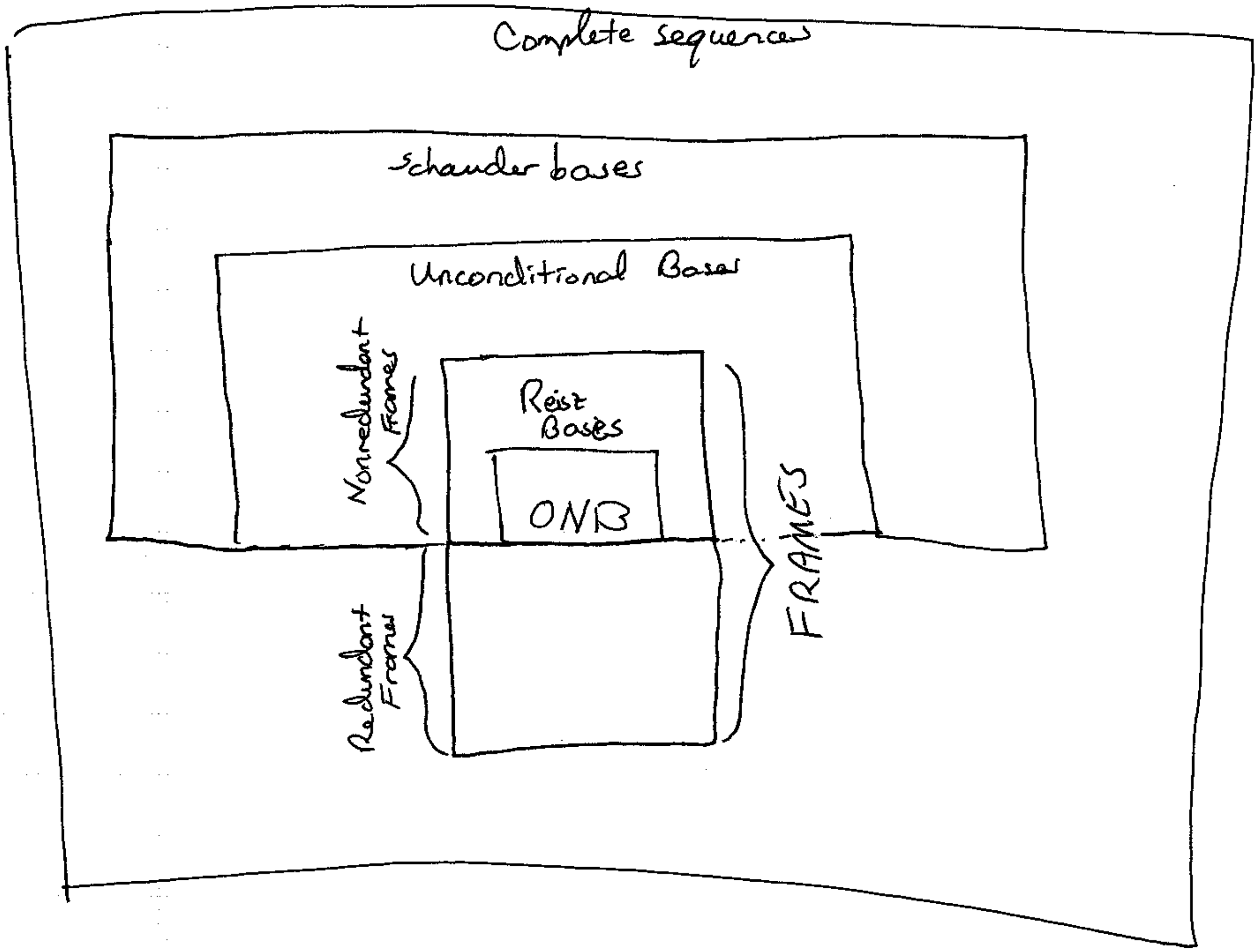
What are a, b? They should be dot products with the dual basis, & in fact:

$$X \cdot \tilde{u} = a\, u \cdot \tilde{u} + b\, v \cdot \tilde{u} = a \cdot 1 + b \cdot 0 = a.$$

&

$$X \cdot \tilde{v} = b$$

The coeff. in the linear comb. are given by dot products with the dual basis elements.



Complete sequences
Schauder bases
Unconditional Bases
Nonredundant Frames
Reisz Bases
ONB
FRAMES
Redundant Frames

6.2 Abstract Frames

A sequence $\{f_n\}_{n\in\mathbb{N}}$ in a Hilbert space H is a <u>frame</u> for H if $\exists\ A, B > 0$ s.t.

$$\forall f \in H, \qquad A\|f\|^2 \le \sum_{n\in\mathbb{N}}' |\langle f, f_n\rangle|^2 \le B\|f\|^2.$$

<u>Notation</u>

a. A, B are called <u>frame bounds</u>. The largest possible A & smallest possible B are the <u>optimal frame bounds</u>.

b. If we can take $A = B$ then the frame is <u>tight</u>.

c. If we can take $A = B = 1$ then it is a <u>Parseval frame</u>.

d. If $\{f_n\}_{n\neq m}$ is not a frame for any $m \in \mathbb{N}$, then the frame is <u>exact</u>. We'll see later that

$$\text{exact frame} \iff \text{frame + Schauder basis}$$
$$\iff \text{Riesz basis}$$

<u>Exercise</u>

a. Prove that all frames are ~~[illegible]~~ complete.

b. Give an example of a complete sequence that is not a frame.

Exercise

a. Let $\{e_n\}_{n\in\mathbb{N}}$, $\{f_n\}_{n\in\mathbb{N}}$ be ONB for a H.S. H.

Prove that $\left\{\frac{e_n}{\sqrt{2}}, \frac{f_n}{\sqrt{2}}\right\}_{n\in\mathbb{N}}$ is a Parseval frame for H that is not exact.

b. Let $\{e_n\}_{n\in\mathbb{N}}$ be an ONB for a H.S. H. Prove that

$$\mathcal{F} = \left\{e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \dots\right\}$$

is a Parseval frame for H. Prove that $\mathcal{F}$ contains a subsequence that is a Schauder basis for H, but this subsequence is not a frame.

Remark

Frames are sequences, so repititions of elements are allowed. Zero elements may also be included.

Note

Even though frames are sequences, we abuse notation & use set notation (e.g., union).

Definition

A sequence $\{f_n\}_{n\in\mathbb{N}}$ is a Bessel sequence for a H.S. H if it has at least an upper frame bound, i.e., $\exists B>0$ s.t.

$$\forall f \in H, \qquad \sum_{n\in\mathbb{N}}' |\langle f, f_n\rangle|^2 \le B\|f\|^2.$$

Exercise

Given a sequence $\mathcal{F}=\{f_n\}_{n\in\mathbb{N}}$ in a H.S. H.

a. Prove that $\{f_n\}_{n\in\mathbb{N}}$ is a Bessel sequence if & only if

$$\forall f\in H, \quad \sum_{n\in\mathbb{N}}' |\langle f, f_n\rangle|^2 < \infty.$$

Hint: U.B.P.

b. Prove that if $\{f_n\}_{n\in\mathbb{N}}$ is Bessel, then the analysis map

$$C : H \longrightarrow \ell^2$$
$$f \longmapsto \{\langle f, f_n\rangle\}_{n\in\mathbb{N}}$$

is bounded, with $\|C\| \le B^{1/2}$.

c. Prove that if $\mathcal{F}$ is a frame then the analysis map is injective.

Exercise

Prove that if $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a frame & $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ is a Bessel sequence, then $\mathcal{F} \cup \mathcal{G} = \{f_n, g_n\}_{n\in\mathbb{N}}$ is a frame.

Lemma

If $\{f_n\}_{n\in\mathbb{N}}$ is a Bessel sequence with Bessel bound B & $c = \{c_n\}_{n\in\mathbb{N}} \in \ell^2$, then $\sum_{n=1}^{\infty} c_n f_n$ converges unconditionally in H, &

$$\Big\| \sum_{n=1}^{\infty} c_n f_n \Big\|^2 \le B \sum_{n=1}^{\infty} |c_n|^2.$$

Proof:

Let $s_N = \sum_{n=1}^{N} c_n f_n$ denote the Nth partial sum, & let $t_N = \sum_{n=1}^{N} |c_n|^2$. Since $\sum_{n=1}^{\infty} |c_n|^2 < \infty$, we have that $\{t_N\}_{N\in\mathbb{N}}$ is a convergent, & hence Cauchy, sequence of real numbers. But we have for any $N > M$ that

$$\|s_N - s_M\|^2 = \sup_{\|f\|=1} |\langle s_N - s_M, f\rangle|^2$$

$$= \sup_{\|f\|=1} \left|\left\langle \sum_{n=M+1}^{N} c_n f_n,\ f\right\rangle\right|^2$$

$$= \sup_{\|f\|=1} \left|\sum_{n=M+1}^{N} c_n \langle f_n, f\rangle\right|^2$$

$$\leq \sup_{\|f\|=1} \left(\sum_{n=M+1}^{N} |c_n|^2\right)\left(\sum_{n=M+1}^{N} |\langle f_n, f\rangle|^2\right)$$

$$\leq \sup_{\|f\|=1} |t_N - t_M| \cdot B\|f\|^2$$

$$= B|t_N - t_M|.$$

Since $\{t_N\}_{N\in\mathbb{N}}$ is Cauchy, we conclude that $\{s_N\}_{N\in\mathbb{N}}$ is Cauchy in H. Hence $\sum_{n=1}^{\infty} c_n f_n = \lim_{N\to\infty} s_N$ exists in H. The same calculation as above (e.g., with $M=0$) shows that $\|s_N\|^2 \leq B t_N$ for each N, & hence

$$\left\| \sum_{n=1}^{\infty}{}' c_n f_n \right\|^2 = \lim_{N\to\infty} \|s_N\|^2 \leq \lim_{N\to\infty} B t_N = B \sum_{n=1}^{\infty}{}' |c_n|^2.$$

If we choose any other ordering of the frame, the same argument applies. That is, if $\sigma : \mathbb{N} \to \mathbb{N}$ is any bijection, & we set

$$s_N^\sigma = \sum_{k=1}^{N}{}' c_{\sigma(n)} f_{\sigma(n)}, \qquad t_N^\sigma = \sum_{n=1}^{N}{}' |c_{\sigma(n)}|^2,$$

then we we have $\|s_N^\sigma - s_M^\sigma\|^2 \leq B\, |t_N^\sigma - t_M^\sigma|$. But since $\sum_{n=1}^{\infty}{}' |c_n|^2$ is a convergent series of <u>nonnegative scalars</u>, it is unconditionally convergent, i.e., $\sum_{n=1}^{\infty}{}' |c_n|^2 = \sum_{n=1}^{\infty}{}' |c_{\sigma(n)}|^2 < \infty$. Hence $\{t_N^\sigma\}_{N\in\mathbb{N}}$ is Cauchy, so $\{s_N^\sigma\}_{N\in\mathbb{N}}$ is Cauchy, hence converges. Thus $\sum_{n=1}^{\infty}{}' c_{\sigma(n)} f_{\sigma(n)}$ converges for every bijection σ, & basic facts on unconditional convergence then imply that the limit is independent of σ, i.e., $\sum_{n=1}^{\infty}{}' c_{\sigma(n)} f_{\sigma(n)} = \sum_{n=1}^{\infty}{}' c_n f_n$. ∎

Definition

If $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ is a Bessel sequence, then

$$R: \ell^2 \longrightarrow H$$
$$c = \{c_n\}_{n \in \mathbb{N}} \longrightarrow \sum_{n=1}^{\infty} c_n f_n$$

is the reconstruction or synthesis mapping.

By the lemma,

$$\|R\| \le B^{1/2}.$$

Exercise

$R = C^*$, i.e.

$$\forall f \in H,\ \forall c \in \ell^2; \quad \langle Cf, c \rangle_{\ell^2} = \langle f, Rc \rangle_H$$

Definition

If $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a frame, then its <u>frame operator</u> is

$$S = RC = C^*C = RR^*.$$

Note that this definition actually makes sense for Bessel sequences, not just frames.

Lemma/Exercise

If $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a Bessel sequence with Bessel bound B, then,

a. S is a bounded map of H into H, with $\|S\| \le B$.

b. S is self-adjoint.

c. $S \geq 0$, i.e., $\langle Sf, f\rangle \geq 0 \ \forall f \in H$.

d. $Sf = \sum_{n=1}^{\infty} \langle f, f_n\rangle f_n$, where the series converges unconditionally in H.

e. $\langle Sf, f\rangle = \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2$

Theorem

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame with frame bounds A, B. Then the following statements hold.

a. S is a continuous bijection of H onto itself.

b. S is positive ($S \geq 0$) & self-adjoint.

c. $AI \leq S \leq BI$.

d. [redacted] $\|Af\| \leq \|Sf\| \leq B\|f\| \quad \forall f \in H$.

Proof:

By definition of frame, we have

$$\underbrace{\langle AIf, f\rangle}_{A\|f\|^2} \leq \underbrace{\langle Sf, f\rangle}_{\sum |\langle f, f_n\rangle|^2} \leq \underbrace{\langle BIf, f\rangle}_{B\|f\|^2}, \quad f \in H$$

Hence $AI \leq S \leq BI$.

The previous exercise established that $\|S\| \leq B$,

so we have $\|Sf\| \leq B\|f\|$. Also,

$$A\|f\|^2 \leq \langle Sf, f\rangle \leq \|Sf\|\,\|f\| \quad \forall f \in H,$$

so $A\|f\| \leq \|Sf\|$ for all f (including $f=0$).

Therefore $A\|f\| \le \|Sf\| \le B\|f\| \quad \forall f$. This establishes that S is a continuous, injective map of H into itself & also that range(S) is closed in H. It therefore remains only to show that range$(S) = H$.

Suppose that $h \in \text{range}(S)^{\perp}$. Then $\langle Sf, h\rangle = 0 \quad \forall f \in H$. Hence $\langle Sh, h\rangle = 0$, which implies $A\|h\|^2 \le \langle Sh, h\rangle = 0$, so $h = 0$. $\blacksquare$

__Exercise__

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame. By expanding $f = SS^{-1}f = S^{-1}Sf$, show that the __frame expansions__

$$f = \sum_{n=1}^{\infty} \langle f, f_n\rangle S^{-1}f_n = \sum_{n=1}^{\infty} \langle f, S^{-1}f_n\rangle f_n$$

converge unconditionally $\forall f \in H$.

__Notation__

We will write $\tilde{f}_n = S^{-1}f_n$, and call $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ the __dual frame__ to $\mathcal{F}$. Of course, we have not yet shown that it *is* a frame — that is our next task.

Theorem

Let $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}}$ be a frame with frame bounds A, B & frame operator S. Let $\tilde{\mathcal{F}} = \{\tilde{f}_n\}_{n \in \mathbb{N}}$.

a. S^{-1} is a continuous bijection of H onto itself.

b. $\frac{1}{B} I \leq S^{-1} \leq \frac{1}{A} I$

c. $\frac{1}{B} \|f\| \leq \|S^{-1}f\| \leq \frac{1}{A} \|f\| \quad \forall f \in H$

d. $\tilde{\mathcal{F}}$ is a frame for H with frame bounds $\frac{1}{B}, \frac{1}{A}$ & frame operator S^{-1}!

Proof:

Given $f \in H$, we have (using the fact that $AI \leq S$)

$$
\begin{aligned}
0 &\leq A \, \|S^{-1}f\|^2 \\
&\leq \langle S(S^{-1}f), S^{-1}f \rangle \\
&= \langle f, S^{-1}f \rangle \\
&\leq \|f\| \, \|S^{-1}f\|.
\end{aligned}
$$

Since $S^{-1}f = 0$ only for $f = 0$, this implies

$$\|S^{-1}f\| \leq \frac{1}{A} \|f\| \qquad \forall f \in H.$$

Consequently,

$$\langle S^{-1}f, f\rangle \leq \|S^{-1}f\| \, \|f\| \leq \frac{1}{A}\|f\|^2 = \langle \tfrac{1}{A}If, f\rangle,$$

so $S^{-1} \leq \frac{1}{A} I$.

To derive the lower inequality, define

$$[f,g] = \langle S^{-1}f, g\rangle, \qquad |||f||| = [f,f]^{1/2} = \langle S^{-1}f, f\rangle^{1/2}.$$

Since S^{-1} is a positive definite operator, $[\cdot,\cdot]$ is an inner product on H (exercise). Consequently, applying the Cauchy-Schwarz inequality w.r.t. both the original inner product & the inner product $[\cdot,\cdot]$, we obtain:

$$
\begin{aligned}
\|f\|^4 &= \langle f, f\rangle^2 \\
&= \langle S^{-1}(Sf), f\rangle^2 \\
&= [Sf, f]^2 \\
&\leq |||Sf|||^2 \, |||f|||^2 \\
&= \langle S^{-1}(Sf), Sf\rangle \langle S^{-1}f, f\rangle
\end{aligned}
$$

$$= \langle f, Sf \rangle \langle S^{-1}f, f \rangle$$

$$\leq \; B\,\|f\|^2 \langle S^{-1}f, f \rangle.$$

Hence $\frac{1}{B}\|f\|^2 \leq \langle S^{-1}f, f\rangle \quad \forall f \in H$, or

$\frac{1}{B} I \leq S^{-1}$. Also $\frac{1}{B}\|f\|^2 \leq \langle S^{-1}f, f\rangle \leq \|S^{-1}f\|\,\|f\|$,

so $\frac{1}{A}\|f\| \leq \|S^{-1}f\| \quad \forall f \in H$.

This completes the proof of statements a – c. Next, note that

$$S^{-1}f = S^{-1}SS^{-1}f$$

$$= S^{-1}\left(\sum_{n=1}^{\infty} \langle S^{-1}f, f_n\rangle f_n\right)$$

$$= \sum_{n=1}^{\infty} \langle f, S^{-1}f_n\rangle\, S^{-1}f_n$$

$$= \sum_{n=1}^{\infty} \langle f, \tilde{f}_n\rangle \tilde{f}_n. \qquad (*)$$

Consequently,

$$\sum_{n=1}^{\infty} |\langle f, \tilde{f}_n\rangle|^2 = \langle S^{-1}f, f\rangle.$$

Combining this with the fact that $\frac{1}{B} I \leq S^{-1} \leq \frac{1}{A} I$

yields the fact that

$$\frac{1}{B}\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, \tilde{f}_n \rangle|^2 \le \frac{1}{A}\|f\|^2$$

Thus $\tilde{\mathcal{F}}$ is a frame with frame bounds $\frac{1}{A}, \frac{1}{B}$.
Finally, (*) shows that S^{-1} is the frame operator for $\tilde{\mathcal{F}}$. $\blacksquare$

Exercise

Let $\{z_n\}_{n=1}^{N}$ denote the N^{th} roots of unity in the complex plane $\mathbb{C}$. Identify $\mathbb{C}$ with $\mathbb{R}^2$, & show that $\{z_n\}_{n=1}^{N}$ is a tight frame for $\mathbb{R}^2$.

Exercise

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a sequence in a H.S. H s.t.

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 \leq B\|f\|^2$$

holds for a dense set of $f \in H$. Show that $\mathcal{F}$ is a frame with frame bounds A, B.

Exercise

Show that if $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a frame for a H.S. H, then H is separable.

Frame Operator

$\{f_n\}$ frame $\quad A\|f\|^2 \le \sum' |\langle f, f_n\rangle|^2 \le B\|f\|^2$

$$Sf = \sum_n{}' \langle f, f_n\rangle f_n \quad \text{converges.}$$

HomeWork: $A = B \implies S = AI$

General: $AI \le S \le BI$ means: $\langle AIf, f\rangle \le \langle Sf, f\rangle \le \langle BIf, f\rangle \ \forall f$

$A\|f\|^2 \qquad \sum |\langle f, f_n\rangle|^2 \qquad B\|f\|^2$

Implies (nontrivially!) that
S is invertible & $\quad \frac{1}{B} I \le S^{-1} \le \frac{1}{A} I$

Proof:

$$\le \frac{B-A}{B+A} < 1$$

$$\left\| I - \frac{2}{A+B} S \right\| < 1$$

exercise: $\left\| f - \frac{2}{A+B} Sf \right\|$

$\frac{2}{A+B} S$ is invertible

$$\left(\frac{2}{A+B} S\right)^{-1} = \sum_{k=0}^{\infty}{}' \left(I - \frac{2}{A+B} S\right)^k$$

$$S^{-1} = \frac{2}{A+B} \sum_{k=0}^{\infty}{}' \left(I - \frac{2}{A+B} S\right)^k$$

converges geometrically

Closer A is to B, better the convergence. $\square$

Frame Expansions $\tilde{f}_n = S^{-1} f_n$

$$f = S^{-1}(Sf) = S^{-1}\left(\sum_{n=1}^{\infty}{}' \langle f, f_n\rangle f_n\right) = \sum_{n=1}^{\infty}{}' \langle f, f_n\rangle S^{-1} f_n$$

$$f = S(S^{-1}f) = \sum_{n=1}^{\infty}{}' \langle S^{-1}f, f_n\rangle f_n = \sum_{n=1}^{\infty}{}' \langle f, S^{-1}f_n\rangle f_n \quad (S^{-1} \text{ is s.a.})$$

So

$$f = \sum{}' \langle f, f_n\rangle \tilde{f}_n = \sum{}' \langle f, \tilde{f}_n\rangle f_n \qquad \tilde{f}_n = S^{-1} f_n$$

When $A = B$, $\tilde{f}_n = \frac{1}{A} f_n$.

Remark $f = \sum{}' c_n f_n \quad (*)$

$$\Longrightarrow \quad \sum{}' |\langle f, \tilde{f}_n\rangle|^2 \leq \sum{}' |c_n|^2.$$

$(\langle f, \tilde{f}_n\rangle)$ has minimal ℓ^2 norm among all (c_n) sat. $(*)$

Frames in Finite Dimensions

Assume $\{v_1, \dots, v_m\}$ span $\mathbb{C}^n$ (so $m \geq n$).

Then

$$\forall v \in \mathbb{C}^n, \quad \exists c_1, \dots, c_m \quad \text{s.t.} \quad v = c_1 v_1 + \dots + c_m v_m$$

but the c_i need not be unique.

Define

$$R = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \qquad C = R^* = R^H = \begin{bmatrix} \text{—} \; v_1^H \; \text{—} \\ \vdots \\ \text{—} \; v_m^H \; \text{—} \end{bmatrix}.$$

Then

$$Rc = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}}_{n \times m} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = c_1 v_1 + \dots + c_m v_m .$$

Reconstruction or synthesis.

$$Cx = \underbrace{\begin{bmatrix} \text{—} \; v_1^H \; \text{—} \\ \vdots \\ \text{—} \; v_m^H \; \text{—} \end{bmatrix}}_{m \times n} x = \begin{bmatrix} v_1^H x \\ \vdots \\ v_m^H x \end{bmatrix} = \begin{bmatrix} v_1 \cdot x \\ \vdots \\ v_m \cdot x \end{bmatrix}$$

Analysis or the coefficient mapping.

The frame operator is $S = RC$

$$Sx = RCx = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \begin{bmatrix} v_1 \cdot x \\ \vdots \\ v_m \cdot x \end{bmatrix} = (v_1 \cdot x) v_1 + \cdots + (v_m \cdot x) v_m$$

Example
If $\{v_1, \dots, v_m\}$ is an ONB for $\mathbb{C}^n$ then $S = I$.

Since $S = C^H C$, it is positive definite:

$$Sx \cdot x = |v_1 \cdot x|^2 + \cdots + |v_m \cdot x|^2 > 0 \quad \text{for } x \neq 0$$

since $v_1, \dots, v_m$ span $\mathbb{C}^n$

S has ONB eigenvectors $\omega_1, \dots, \omega_n$
evalues $\lambda_1 > \cdots > \lambda_n > 0$

Write $x = (\omega_1 \cdot x)\omega_1 + \cdots + (\omega_n \cdot x)\omega_n = \sum_i (\omega_i \cdot x)\omega_i$

$\|x\|^2 = \sum_i |\omega_i \cdot x|^2$

$$Sx \cdot x = \sum (\omega_i \cdot x) S\omega_i \cdot x$$

$$= \sum_{i=1}^{n} \lambda_i \, |\omega_i \cdot x|^2$$

$\sum_i |v_i \cdot x|^2$

$$\lambda_n \|x\|^2 \le Sx \cdot x \le \lambda_1 \|x\|^2$$ Frame ineq.!

6.3 Example: Frames of Exponentials

Consider the Hilbert space $L^2[0,1]$. Let $\mathbb{T} = [0,1]$ under addition mod 1; then $\mathbb{T}$ is topological a circle or 1-dimensional torus. We can identify $L^2(\mathbb{T}) = L^2[0,1]$ with

$$\left\{ f : \mathbb{R} \to \mathbb{C} \;:\; f \text{ is 1-periodic } \& \int_0^1 |f(t)|^2\,dt < \infty \right\}$$

With this identification, the interval $[0,1]$ can be replaced by any interval $[T, T+1]$. For example, $[-\frac{1}{2}, \frac{1}{2}]$ gives a domain symmetric about the origin.

Another useful identification is with

$$\left\{ f : \mathbb{R} \to \mathbb{C} \;:\; \operatorname{supp}(f) \subseteq [0,1] \;\&\; \int_0^1 |f(t)|^2\,dt < \infty \right\}$$

$$= \left\{ f \in L^2(\mathbb{R}) : \operatorname{supp}(f) \subseteq [0,1] \right\}.$$

Both identifications are useful, depending on context.

Basic Fact

If we set $e_n(t) = e^{2\pi i n t}$, then $\{e_n\}_{n\in\mathbb{Z}}$

is an ONB for $L^2[0,1]$.

Definition

The Fourier transform of $f \in L^2[0,1]$ is the sequence $\hat{f} \in \ell^2(\mathbb{Z})$ defined by

$$\hat{f}(n) = \langle f, e_n \rangle = \int_0^1 f(t)\, e^{-2\pi i n t}\, dt, \quad n \in \mathbb{Z}.$$

The Fourier transform of $c = \{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ is the function $\hat{c} \in L^2[0,1]$ given by

$$\hat{c}(\xi) = \sum_{n\in\mathbb{Z}}' c_n e^{+2\pi i n \xi} = \sum_{n\in\mathbb{Z}}' c_n e_n(\xi)$$

The series converges in L^2-norm because $\{e_n\}_{n\in\mathbb{Z}}$ is an ONB for $L^2[0,1]$.

Exercise:

State these operators in terms of the analysis & synthesis operators associated with the frame $\{e_n\}_{n\in\mathbb{Z}}$.

Remark

$\mathbb{T}$ & $\mathbb{Z}$ are dual groups under the F.T.

On the real line, the dual group to $\mathbb{R}$ is $\mathbb{R}$ itself.

The dual group $\widehat{G}$ of a locally compact abelian group G consists of all the characters χ on G, i.e., the continuous functions $\chi : G \to \mathbb{C}$ satisfying

$$|\chi(x)| = 1 \quad \forall x \in G$$

$$\chi(x+y) = \chi(x)\chi(y) \quad \forall x,y \in G.$$

Remark

Much of the theory of the F.T. on $\mathbb{T}$ (also referred to as Fourier series) is similar to that on $\mathbb{R}$. Sometimes there are simplifications due to the fact that $\mathbb{T}$ has finite measure, so in particular,

$$L^\infty[0,1] \subseteq L^2[0,1] \subseteq L^1[0,1].$$

Even so, questions such as pointwise convergence issues can be extremely difficult.

Inversion Formula

If $f \in L^1[0,1]$ & $\hat{f} \in \ell^1(\mathbb{Z})$, then

$$f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e_n(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n t}$$

holds pointwise.

Approximate identity arguments, in particular using the Fejer kernel, give the following for general $f \in L^1(\mathbb{T})$.

Theorem

Given $f \in L^1[0,1]$, define

$$f_N = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) \hat{f}(n)\, e_n .$$

Then $f_N \longrightarrow f$ in $L^1[0,1]$ as $N \to \infty$.

Corollary: Uniqueness Theorem.

$\{e_n\}_{n\in\mathbb{Z}}$ is complete in $L^1[0,1]$. In particular,

$$f \in L^1[0,1],\ \hat{f}(n) = 0\ \forall n \in \mathbb{Z} \implies f = 0.$$

Proof:

The preceding theorem implies that the finite linear span of $\{e_n\}_{n\in\mathbb{Z}}$ is dense in $L^1[0,1]$. $\blacksquare$

Remark

The theorem does not imply that $\{e_n\}_{n\in\mathbb{Z}}$ is a basis for $L^1[0,1]$. For, it only yields expansions of the form

$$f = \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f,N)\, e_n$$

with coefficients $c_n(f,N)$ that depend on both f & N.