Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

Definition.
\[ C^m_b(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : f, f', \ldots, f^{(m)} \text{ exist and are continuous and bounded} \}, \]
\[ C^m_c(\mathbb{R}) = \{ f \in C^m_b(\mathbb{R}) : f \text{ is compactly supported} \}. \]

1. Prove Young’s Inequality: If \( 1 \leq p, q \leq \infty \) and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \), then
\[ \forall f \in L^p(\mathbb{R}), \quad \forall g \in L^q(\mathbb{R}), \quad \| f \ast g \|_r \leq \| f \|_p \| g \|_q. \]

2. a. Prove that if \( f \in L^1(\mathbb{R}) \) and \( g \in C^m_c(\mathbb{R}) \) then \( f \ast g \in C^m_b(\mathbb{R}) \), and furthermore
\[ D^j(f \ast g) = f \ast D^j g, \quad j = 1, \ldots, m. \]

b. Prove that if we assume in addition that \( f \) has compact support, then \( f \ast g \in C^m_c(\mathbb{R}) \).

3. Let \( M \) be a closed subspace of \( L^1(\mathbb{R}) \). Prove that the following two statements are equivalent.
   a. \( M \) is translation-invariant, i.e., if \( f \in M \) and \( a \in \mathbb{R} \) then \( T_a f \in M \).
   b. \( M \) is an ideal in \( L^1(\mathbb{R}) \), i.e., if \( f \in M \) and \( g \in L^1(\mathbb{R}) \) then \( f \ast g \in M \).

Hint for a \( \Rightarrow \) b: Hahn–Banach and \( (L^1)^* = L^\infty \).

4. Let \( \{ k_\lambda \}_{\lambda > 0} \) be an approximate identity.
   a. Prove that if \( f \in C_0(\mathbb{R}) \) then \( f \ast k_\lambda \) converges to \( f \) uniformly as \( \lambda \to \infty \).

   b. Fix \( 1 < p < \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Prove that if \( f \in L^p(\mathbb{R}) \) and \( g \in L^{p'}(\mathbb{R}) \) then \( f \ast g \in C_0(\mathbb{R}) \).
   Hint: Let \( f_n, g_n \in C_c(\mathbb{R}) \) converge to \( f, g \).
5. Define $f_k = \chi_{[-1,1]} \ast \chi_{[-k,k]}$.

a. Find an explicit formula for $f_k$ and show that $\|f_k\|_\infty = 2$.

b. Show that $\lim_{k \to \infty} \|\hat{f}_k\|_1 = \infty$.
Hints: Take the Fourier transform of $f_k$ and make the change of variables $\eta = 2\pi k \xi$.

c. Show that $A(\mathbb{R}) = \text{range}(\mathcal{F}) \subseteq C_0(\mathbb{R})$.
Hints: You can assume that $\mathcal{F}$ is injective. Consider the Inverse Mapping Theorem and the fact that $f_k$, $\hat{f}_k \in L^1(\mathbb{R})$, so the Inversion Theorem applies.

d. Show that $C^2(\mathbb{R}) \subset A(\mathbb{R})$.

e. Show that $A(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

6. Suppose that $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and that
$$f(0^+) = \lim_{t \to 0^+} f(t), \quad f(0^-) = \lim_{t \to 0^-} f(t)$$
both exist, but $f(0^-) \neq f(0^+)$.

Let $G(x) = e^{-\pi x^2}$ be the Gauss kernel, and set $G_\lambda(x) = \lambda G(\lambda x)$.

a. Prove that
$$\lim_{\lambda \to \infty} (f * G_\lambda)(0) = \lim_{\lambda \to \infty} \langle f, G_\lambda \rangle = \frac{f(0^+) + f(0^-)}{2}.$$ 
Note: Although $f$ need not be in $L^2(\mathbb{R})$, the “inner product” $\langle f, G_\lambda \rangle = \int f(x) G_\lambda(x) \, dx$ is well-defined since $G_\lambda$ is bounded.

b. Prove that $(\hat{f} \hat{G_\lambda})^\vee = f * G_\lambda$.

c. Prove that $\langle \hat{f}, \hat{G_\lambda} \rangle = \langle f, G_\lambda \rangle$.

d. Suppose $\hat{f}(\xi)$ is real for all $\xi$, and that $\hat{f}(\xi) > 0$ for all $|\xi| > R$. Prove that
$$\int_{|\xi| > R} \hat{f}(\xi) \, d\xi < \infty.$$ 
Hint: Consider $\lim \inf_{\lambda \to \infty} \int_{|\xi| > R} \hat{f}(\xi) \hat{G_\lambda}(\xi) \, d\xi$.

e. Prove that there is no $f$ satisfying the assumptions on $\hat{f}$ in part d. Can you show that if $\hat{f}$ is real then it must change sign infinitely many times?