MATH 7337 TAKE-HOME FINAL EXAM DUE: 3:00 p.m. December 11, 2008

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. We know that $L^1(\mathbb{R})$ embeds isometrically into $M_b(\mathbb{R})$ by identifying $g \in L^1(\mathbb{R})$ with the measure

$$\mu_g(E) \;=\; \int_E g(x)\,dx, \qquad \text{Borel } E \subseteq \mathbb{R}.$$

Prove that $L^1(\mathbb{R})$ is a closed ideal in $M_b(\mathbb{R})$ with respect to convolution.

2. Prove that δ' (distributional derivative of δ) does not belong to $M_b(\mathbb{R})$.

3. Let $g \in L^1_{\mathrm{loc}}(\mathbb{R})$, and identify g with the distribution in $\mathcal{D}'(\mathbb{R})$ that it defines. Show that if $g \in \mathcal{E}'(\mathbb{R})$, then there exists a compact set K such that $g(x) = 0$ for a.e. $x \notin K$.

4. Show that if $\mu \in \mathcal{D}'(\mathbb{R})$ and the distributional derivative of μ satisfies $D\mu = 0$, then μ is a constant function.

Hint: Fix $k \in C_c^\infty(\mathbb{R})$ with $\int k = 1$. Show that every $f \in C_c^\infty(\mathbb{R})$ can be written uniquely as $f = c_f k + g$ where $c_f \in \mathbb{C}$ and $g = h'$ for some $h \in C_c^\infty(\mathbb{R})$.

5. Let $\phi(x) = e^{-\pi x^2}$ be the Gaussian function. The *modulation space* $M^1(\mathbb{R})$ is

$$M^1(\mathbb{R}) \;=\; \{f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{M^1} < \infty\},$$

where

$$\|f\|_{M^1} \;=\; \|V_\phi f\|_1 \;=\; \iint |V_\phi f(x,\xi)|\,dx\,d\xi.$$

Note: You can assume the following facts without proof.

i. We proved that if $f \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$ then $V_g f \in C^\infty(\mathbb{R})$ with polynomial growth at infinity.

ii. It can be shown that $(M^1(\mathbb{R}), \|\cdot\|_{M^1})$ is a Banach space.

iii. We proved that $V_g f \in \mathcal{S}(\mathbb{R}^2)$ whenever $f,\, g \in \mathcal{S}(\mathbb{R})$, so $\mathcal{S}(\mathbb{R}) \subseteq M^1(\mathbb{R})$.

iv. Although this is not needed for this problem, it can be shown that $M^1(\mathbb{R}) \subseteq L^2(\mathbb{R})$.

v. A consequence of the Inversion Formula for the STFT is that

$$\forall\, f \in \mathcal{S}'(\mathbb{R}), \quad \forall\, g, \gamma, h \in \mathcal{S}(\mathbb{R}), \quad |V_h f(x,\xi)| \;\le\; \frac{1}{|\langle \gamma, g\rangle|} \big(|V_g f| * |V_h \gamma|\big)(x,\xi). \tag{1}$$

(a) Prove that if g is any nonzero function in $\mathcal{S}(\mathbb{R})$, then

$$M^1(\mathbb{R}) \;=\; \{f \in \mathcal{S}'(\mathbb{R}) : \|V_g f\|_{L^1} < \infty\},$$

and that

$$|\!|\!|f|\!|\!| \;=\; \|V_g f\|_1 \;=\; \iint |V_g f(x,\xi)|\, dx\, d\xi, \qquad f \in M^1(\mathbb{R}),$$

is an equivalent norm for $M^1(\mathbb{R})$.

(b) Prove that $M^1(\mathbb{R})$ is invariant under the Fourier transform.

6. The *Wiener amalgam space* $W(L^\infty, \ell^1)$ is

$$W(L^\infty, \ell^1) \;=\; \{f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \|f\|_{W(L^\infty,\ell^1)} < \infty\},$$

where

$$\|f\|_{W(L^\infty,\ell^1)} \;=\; \sum_{k\in\mathbb{Z}} \|f\,\chi_{[k,k+1]}\|_\infty.$$

It can be shown that $W(L^\infty, \ell^1)$ is a Banach space. Let $g \in C_c^\infty(\mathbb{R})$ be such that $0 \le g \le 1$ everywhere and $g = 1$ on $[-1,1]$. Show that if $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $x \in [0,1]$, then

$$\|f\,\chi_{[k,k+1]}\|_\infty \;\le\; \|f \cdot T_{k+x}\bar{g}\|_\infty \;\le\; \int |V_g f(x+k,\xi)|\, d\xi,$$

and use this to prove that

$$\|f\|_{W(L^\infty,\ell^1)} \;\le\; \|f\|_{M^1}.$$

Consequently,

$$M^1(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}) \;\subseteq\; W(L^\infty, \ell^1) \;\subseteq\; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

Show that

$$M^1(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R}) \;\subseteq\; L^1(\mathbb{R}) \cap A(\mathbb{R}).$$

Remark: It can be shown that $M^1(\mathbb{R}) \subseteq L^1_{\mathrm{loc}}(\mathbb{R})$, so the restriction in this problem to $L^1_{\mathrm{loc}}(\mathbb{R})$ is not needed.