MATH 7337 HOMEWORK #4 Due date: November 3, 2008

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. (a) Prove that if $g \in \mathrm{BV}(\mathbb{R}) \cap \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$, then $Dg = g'$ in the sense of distributions.

(b) Prove that $L^1(\mathbb{R}) = \big\{Dg : g \in \mathrm{BV}(\mathbb{R}) \cap \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})\big\}$.

Hint: The variation of g over $[a,b]$ is

$$V_{[a,b]}(x) \;=\; \sup\Biggl\{\sum_{j=1}^{N} |g(x_j) - g(x_{j-1})| : a = x_0 < x_1 < \cdots < x_N = x\Biggr\}, \qquad x \in [a,b].$$

The variation function $V_{[a,b]}$ is monotone increasing and finite on $[a,b]$, and by Corollary 7.23 in Wheeden and Zygmund we have

$$V'_{[a,b]}(x) \;=\; |g'(x)|, \qquad x \in [a,b].$$

2. (a) Let $\mu \in \mathcal{D}'(\mathbb{R})$ and $\theta \in C^\infty(\mathbb{R})$ be given. Show that $\langle f, \theta\mu\rangle = \langle f\bar{\theta}, \mu\rangle$ defines a distribution $\theta\mu \in \mathcal{D}'(\mathbb{R})$.

(b) Show that if $\mu \in \mathcal{E}'(\mathbb{R})$ and $\theta \in C^\infty(\mathbb{R})$, then $\theta\mu \in \mathcal{E}'(\mathbb{R})$.

3. Prove that if at least one of $\mu, \nu \in \mathcal{D}'(\mathbb{R})$ has compact support, then $f*(\mu*\nu) = (f*\mu)*\nu$ for every $f \in C_c^\infty(\mathbb{R})$.

4. Define $\mathrm{pv}(\frac{1}{x})$ by

$$\langle f, \mathrm{pv}(\tfrac{1}{x})\rangle \;=\; \lim_{T\to\infty} \int_{\frac{1}{T}<|x|<T} \frac{f(x)}{x}\,dx, \qquad f \in \mathcal{S}(\mathbb{R}). \tag{1}$$

Prove that $\mathrm{pv}(\frac{1}{x}) \in \mathcal{S}'(\mathbb{R})$.

Hint: Break the integral into $\frac{1}{T} < |x| < 1$ and $1 < |x| < T$.

5. Compute the distributional Fourier transforms of $e_\xi(x) = e^{2\pi i \xi x}$ and $D^n\delta$.