Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

Definitions/Assumptions. (a) The topology on $C_c(\mathbb{R})$ is the inductive limit topology. That is, $f_k \to f$ in $C_c(\mathbb{R})$ if there exists a single compact set K such that $\operatorname{supp}(f_k) \subseteq K$ for every k, and $\|f - f_k\|_\infty \to 0$ as $k \to \infty$.

(b) We let $M_r(\mathbb{R})$ denote the set of all signed Radon measures on $\mathbb{R}$ (which equals the set of all signed locally finite Borel measures on $\mathbb{R}$). Note that the measures in $M_r(\mathbb{R})$ need not be bounded.

(c) The topology on $C_0(\mathbb{R})$ is the uniform norm topology. The topology on $M_b(\mathbb{R})$ is the topology induced from the norm $\|\mu\| = |\mu|(\mathbb{R})$. A Riesz Representation Theorem asserts that $M_b(\mathbb{R}) \cong C_0(\mathbb{R})^*$.

1. (a) Show directly that if $\nu \in M_r(\mathbb{R})$, then $T_\nu \colon C_c(\mathbb{R}) \to \mathbb{C}$ defined by

$$\langle f, T_\nu \rangle \;=\; \langle f, \nu \rangle \;=\; \int f(x)\, d\bar{\nu}(x), \qquad f \in C_c(\mathbb{R}),$$

is a linear functional on $C_c(\mathbb{R})$ that is continuous with respect to the inductive limit topology on $C_c(\mathbb{R})$.

Remark: This shows that $M_r(\mathbb{R}) \subseteq C_c(\mathbb{R})^*$, in the sense of identifying ν with T_ν. A Riesz Representation Theorem states that every positive linear functional on $C_c(\mathbb{R})$ is induced from a positive Radon measure.

(b) Show that the identification of part (a) does not yield an embedding of $M_r(\mathbb{R})$ into $(C_c(\mathbb{R}), \|\cdot\|_\infty)^*$.

Hint: Consider the delta train $\mu = \sum_{n\in\mathbb{Z}} \delta_n$.

2. Prove that $M_b(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$, in the sense that there is a continuous antilinear injection of $M_b(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$.

Note: The topology on $\mathcal{S}'(\mathbb{R})$ is the weak* topology, so to prove the continuity of a map $T \colon M_b(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, you must suppose that $\{\nu_i\}_{i\in I}$ is any net in $M_b(\mathbb{R})$ such that $\nu_i \to \nu$ in the norm of $M_b(\mathbb{R})$, and show that $T(\nu_i) \xrightarrow{\text{w}^*} T(\nu)$.

3. Let $\delta_n \in M_b(\mathbb{R})$ denote the point mass at n measure, and set

$$\mu_n \;=\; \frac{1}{n}\,(\delta_1 + \cdots + \delta_n).$$

(a) Show that $\mu_n \xrightarrow{\text{w*}} 0$.

(b) Show that $\varphi(\xi) = \lim_{n\to\infty} \widehat{\mu_n}(\xi)$ exists for every ξ, but φ is not continuous. In particular, $\widehat{\mu_n} \not\to \hat{0}$, i.e., weak* convergence does not imply convergence of the corresponding Fourier transforms to the expected limit even if the Fourier transforms do converge.

(c) Suppose that $\nu_n,\ \nu \in M_b(\mathbb{R})$ and $\nu_n \xrightarrow{\text{w*}} \nu$. Show that if $\varphi \in C(\mathbb{R})$ and $\widehat{\nu_n}(\xi) \to \varphi(\xi)$ pointwise, then $\varphi = \hat{\nu}$. Thus, weak* convergence combined with convergence of the corresponding Fourier transforms to a *continuous* function implies that the limit is the expected function.

Hints: Weak* convergent sequences are bounded (this is a consequence of the Uniform Boundedness Principle). Show that $\langle \widehat{f}, \varphi\rangle = \langle f, \nu\rangle$ for all $f \in \mathcal{F}C_c^\infty(\mathbb{R})$.

4. Let $\varphi \in C(\mathbb{R})$ be given. Let w be the Fejér kernel, and set $w_\lambda(x) = \lambda w(\lambda x)$. Define $f_\lambda = (\varphi\,\widehat{w_\lambda})^\vee$, i.e.,

$$f_\lambda(x) \;=\; \int_{-\lambda}^{\lambda} \varphi(\xi)\left(1 - \frac{|\xi|}{\lambda}\right) e^{2\pi i\xi x}\,d\xi.$$

Prove that the following statements are equivalent.

(a) $\varphi = \hat{\nu}$ for some $\nu \in M_b(\mathbb{R})$.

(b) $f_\lambda \in L^1(\mathbb{R})$ for every λ, and $\sup_\lambda \|f_\lambda\|_1 < \infty$.

Hints for (b) $\Rightarrow$ (a): Define $\nu_n = f_n\,dx \in M_b(\mathbb{R})$. Apply Alaoglu's Theorem to find a weak* convergent subsequence.