3. Distributions & the Fourier Transform

3.1 Motivation & the δ functional

Recall that $L^1(\mathbb{R})$ is a Banach algebra *without identity* under convolution: $\nexists\, \delta \in L^1(\mathbb{R})$ s.t. $\delta * f = f \quad \forall f \in L^1(\mathbb{R})$. For, if there was such a function δ, then it would satisfy $\hat{\delta}\,\hat{f} = \hat{f} \quad \forall f \in L^1(\mathbb{R})$, which implies $\hat{\delta} \equiv 1$, contradicting the Riemann-Lebesgue lemma.

But suppose for the moment that such a δ existed. Then we would have (set $f^{*}(x) = f(-x)$):

$$\begin{aligned} f(0) = f^{*}(0) &= \left(f^{*} * \delta\right)(0) \\ &= \int f^{*}(0-t)\,\delta(t)\,dt \\ &= \int f(t)\,\delta(t)\,dt. \end{aligned}$$

While there is no function $\delta \in L^1(\mathbb{R})$ for which this true, we can consider

$$\text{"} \int f(t)\,\delta(t)\,dt = f(0) \text{"}$$

as a mapping taking the function f to the value f(0).

In order to make point values well-defined, we should require f to be continuous.

<u>Lemma</u>

Define

$$\begin{aligned} \delta : C_b(\mathbb{R}) &\longrightarrow \mathbb{C} \\ f &\longmapsto f(0) \end{aligned}$$

Then δ is a continuous linear functional on $C_b(\mathbb{R})$

<u>Remark.</u> A "functional" on a space is a scalar-valued function whose domain is that space.

<u>Proof:</u>

δ is linear since

$$\begin{aligned} \delta(\alpha f + \beta g) &= (\alpha f + \beta g)(0) \\ &= \alpha f(0) + \beta g(0) \\ &= \alpha\,\delta(f) + \beta\,\delta(g). \end{aligned}$$

Since $C_b(\mathbb{R})$ is a Banach space, to show δ is continuous we just have to show it is bounded:

$$\|\delta\| = \sup_{\|f\|_\infty = 1} |\delta(f)| \qquad \text{(norm on } \mathbb{C} \text{ is absolute value)}$$

$$= \sup_{\|f\|_\infty = 1} |f(0)|$$

$$\leq \sup_{\|f\|_\infty = 1} \|f\|_\infty = 1. \blacksquare$$

Note: In fact, $\|\delta\| = 1$ since $\exists f \in C_b(\mathbb{R})$ s.t. $\|f\|_\infty = 1$ & $|f(0)| = 1$, e.g., the tent function.

Thus δ is not a function on $\mathbb{R}$ but rather a functional that maps functions to scalars.

Exercise

Although "$\int f(t)\delta(t)\,dt = f(0)$" does not make literal sense, we may hope that an approximate identity $\{k_\lambda\}_{\lambda>0}$ will give an approximate version of this statement. Prove that

$$\forall f \in C_b(\mathbb{R}), \quad \lim_{\lambda \to \infty} \int f(x) k_\lambda(x)\,dx = \delta(f) = f(0).$$

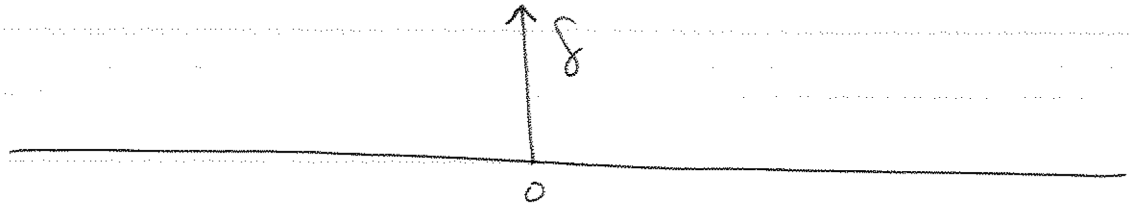
Remark

δ is known by many names, including:

- the point mass at zero measure
- the Dirac measure at $x=0$
- the delta measure at $x=0$
- the "Dirac delta function"
- the delta distribution

and probably many more. We will *see* in Chapter 4 why δ is not only a distribution but a bounded measure on $\mathbb{R}$.

Although not literally correct, we often depict δ pictorially as a function which is zero for all $x \neq 0$ yet satisfies $\int \delta = 1$:



But no such function exists, since such a function would be *zero a.e.* & hence would have integral zero. Still, in spirit much of the picture is accurate – δ is a *functional*, $\operatorname{supp}(\delta) = \{0\}$ in the sense that if $\operatorname{supp}(f) \subseteq \mathbb{R} \setminus \{0\}$ then $\langle f, \delta \rangle = 0$, etc. We will explain these issues for δ & more general functionals in the coming sections.

Definition

If X is a normed linear space then its dual space X^* or X' is the space of all continuous linear functionals on X:

$$X^* = \{ T : X \to \mathbb{C} \ : \ T \text{ is linear \& continuous} \}.$$

Exercise: X^* is a Banach space (even if X is not) under the operator norm:

$$\| T \| = \sup_{\|f\|_X = 1} | T(f) |.$$

Example

Thus $\delta \in C_b(\mathbb{R})^*$, & similarly $\delta \in C_0(\mathbb{R})^*$.

The Riesz Representation Theorem asserts that $C_0(\mathbb{R})^*$ can be identified with $M_b(\mathbb{R})$, the space of bounded Radon measures on $\mathbb{R}$.

We will return to this in ~~[illegible]~~ more detail later.

Examples/Exercises

Fix $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Consider $X = L^p(\mathbb{R})$.

For each $g \in L^{p'}(\mathbb{R})$, define $T_g : L^p(\mathbb{R}) \to \mathbb{C}$ by

$$T_g(f) = \int f(x)\, g(x)\, dx, \qquad f \in L^p(\mathbb{R}).$$

This is well-defined by Hölder's Inequality.

a. Show $T_g \in L^p(\mathbb{R})^*$, with $\|T_g\| \le \|g\|_{p'}$.

b. Show $\|T_g\| = \|g\|_{p'}$.

Hint: Consider $f = \dfrac{|g|^{p-1}}{\|g\|_{p'}^{p'-1}}$

c. Define

$$\begin{aligned} \mathcal{U} : L^{p'}(\mathbb{R}) &\longrightarrow L^p(\mathbb{R})^* \\ g &\longmapsto T_g \end{aligned}$$

Prove that $\mathcal{U}$ is an injective linear isometry.

Conclude that $\mathcal{U}$ has closed range.

d. Prove that $\mathcal{U}$ is surjective if $1 \le p < \infty$.

Notation abuse

In light of the preceding exercise we usually identify $L^{p'}(\mathbb{R})$ with its image in $L^p(\mathbb{R})^*$ under $\mathcal{U}$. Thus we write

$$L^{p'}(\mathbb{R}) \;\text{[illegible]}= L^p(\mathbb{R})^*, \qquad 1 \le p < \infty$$

$$L^1(\mathbb{R}) \subsetneq L^\infty(\mathbb{R})^*, \qquad p = \infty.$$

More notation

In order to make the notation more consistent with the inner product on $L^2(\mathbb{R})$, we often use the alternative identification

$$\tilde{T}_g(f) = T_{\bar{g}}(f) = \int f(x)\,\overline{g(x)}\,dx = \langle f, g \rangle$$

Notation abuse: only an inner product when $p = 2$

$$\begin{aligned} \tilde{\mathcal{U}} : L^{p'}(\mathbb{R}) &\longrightarrow L^p(\mathbb{R})^* \\ g &\longmapsto \tilde{T}_g \end{aligned}$$

This is often much more notationally convenient; the only complication is that $\tilde{\mathcal{U}}$ is <u>anti-liner</u>, ie.,

$$\tilde{\mathcal{U}}(\alpha g + \beta h) = \bar{\alpha}\,\tilde{\mathcal{U}}(g) + \bar{\beta}\,\tilde{\mathcal{U}}(h)$$

Yet more notation

As in the last remark, if $g \in L^{p'}(\mathbb{R})$ then

$$f \mapsto \langle f, g \rangle = \int f(x)\,\overline{g(x)}\,dx$$

defines a bounded linear functional on $L^p(\mathbb{R})$, (as does

$$f \longmapsto \int f(x)\,g(x)\,dx)$$

Because of this we often use a kind of "generalized inner product" notation to represent continuous linear functionals, e.g., for $\delta \in C_b(\mathbb{R})^*$ we may write

$$\langle f, \delta \rangle = f(0)$$

with the understanding that this "inner product" is anti-linear in the 2nd variable:

$$\langle f, \alpha\delta \rangle = \bar{\alpha}\,\langle f, \delta \rangle = \bar{\alpha}\, f(0).$$

In contrast, the notation $\delta(f)$ is *linear* in δ, i.e.

$$(\alpha\delta)(f) = \alpha\,\delta(f) = \alpha f(0).$$

This technical point is unpleasant, but usually only a book keeping annoyance.

Exercise

a. Using T_g (or $\tilde{T}_g$) as defined in a previous example, show that

$$L^1(\mathbb{R}) \subseteq \text{[redacted]}\, C_b(\mathbb{R})^* \qquad (*)$$

Make this inclusion precise, i.e., [redacted] make an explicit isometric embedding of $L^1(\mathbb{R})$ into $C_b(\mathbb{R})^*$.

b. Show that the inclusion in $(*)$ is proper.

c. Show that $C_b(\mathbb{R})^* \subseteq C_0(\mathbb{R})^*$.

Remark: The fact that the embedding

$$\begin{aligned} C_0(\mathbb{R}) &\longrightarrow C_b(\mathbb{R}) \\ f &\longmapsto f \end{aligned}$$

is continuous is important, [redacted] i.e., there is a connection between the topology on $C_0(\mathbb{R})$ & the topology on $C_b(\mathbb{R})$.

Exercise

Let X be a Banach space. For each $f \in X$ define

$$\begin{aligned} f^{**} : X^* &\longrightarrow \mathbb{C} \\ T &\longmapsto \langle f, T \rangle. \end{aligned}$$

That is, $\langle T, f^{**} \rangle = \langle f, T \rangle$, or $f^{**}(T) = T(f)$.

Show that $f \longmapsto f^{**}$ is an injective isometry of X into X^{**}, i.e., $X \subseteq X^{**}$.

Definition

A Banach space X is reflexive if $X = X^{**}$, i.e., the mapping $f \longmapsto f^{**}$ given above is surjective.

Exercise

Prove that $L^p(\mathbb{R})$ is reflexive for $1 < p < \infty$.

Exercise

Define $c_0 = \{ x = (x_n)_{n=1}^{\infty} : x_n \in \mathbb{C}, \lim_{n \to \infty} x_n = 0 \}$.

Prove that $c_0^* = \ell^1$ (& thus $c_0^{**} = \ell^\infty$).

Remark: This exercise is related to bounded measures on $\mathbb{N}$.

Theorem (see Functional Analysis notes)
If X is a normed linear space, then

$$X^* \text{ separable} \implies X \text{ separable.}$$

Consequently, if X is a reflexive Banach space then

$$X \text{ separable} \iff X^* \text{ separable.}$$

Exercise
Use this to show that $L^1(\mathbb{R})$ & $L^\infty(\mathbb{R})$ are not reflexive.

Recall that an earlier exercise showed that if $\{k_\lambda\}_{\lambda>0}$ is an approximate identity then

$$\forall f \in C_b(\mathbb{R}), \qquad \int f(x)\,k_\lambda(x)\,dx \longrightarrow f(0)$$

We now give some notation which allows us to interpret this as a statement that k_λ converges to δ in a certain sense.

<u>Definition</u>

Let X be a Banach space.

a. Given $f_n, f \in X$, we say that <u>f_n converges weakly to f</u> $(f_n \xrightarrow{\omega} f)$ if

$$\forall T \in X^*, \qquad \langle f_n, T\rangle \longrightarrow \langle f, T\rangle$$

b. Given $T_n, T \in X^*$, we say that <u>T_n converges weak* to T</u> $(T_n \xrightarrow{\omega^*} T)$ if

$$\forall f \in X, \qquad \langle f, T_n\rangle \longrightarrow \langle f, T\rangle.$$

Exercise

Let $\{k_\lambda\}_{\lambda>0}$ be an approximate identity. Observe that $K_\lambda \in L^1(\mathbb{R}) \subseteq C_b(\mathbb{R})^*$, & $\delta \in C_b(\mathbb{R})^*$.

Prove that
$$K_\lambda \xrightarrow{w^*} \delta.$$

Exercise

Let $f_n, f \in C_0(\mathbb{R})$. Show that

$$f_n \xrightarrow{w} f \iff \forall x \in \mathbb{R},\ f_n(x) \to f(x)\ \&\ \sup_n \|f\|_\infty < \infty.$$

Hint: Uniform Boundedness Principle.

Exercise

a. Suppose that $f_n, f \in L^1(\mathbb{R})$, $f_n \to f$ pointwise a.e., and $\exists\, g \in L^1(\mathbb{R})$ s.t. $|f_n(x)| \leq g(x)$ a.e. Show $f_n \xrightarrow{w} f$.

b. Suppose $\varphi_n, \varphi \in L^\infty(\mathbb{R})$, $\varphi_n \to \varphi$ pointwise a.e., and $\sup_n \|\varphi_n\|_\infty < \infty$. Show $\varphi_n \xrightarrow{w^*} \varphi$.

c. Let $e_n(x) = e^{2\pi i n x} \in L^\infty(\mathbb{R})$. Show $e_n \xrightarrow{w^*} 0$.

<u>Exercise</u> Let X be a Banach space.

a. All weakly convergent sequences in X are bounded.

b. All weak* convergent sequences in X^* are bounded.

Hint: Uniform Boundedness Principle.

Remark

Since $L^1(\mathbb{R}) \subseteq C_b(\mathbb{R})^*$, any definition that we make for objects in $C_b(\mathbb{R})^*$ should have an appropriate interpretation for the particular case of functions in $L^1(\mathbb{R})$. Conversely, definitions that apply to functions in $L^1(\mathbb{R})$ may extend naturally to all of $C_b(\mathbb{R})^*$.

Example: Function notation for a functional.

If $g \in L^1(\mathbb{R})$ then the translation of g by $a \in \mathbb{R}$ is $T_a g(x) = g(x-a)$. Considered as an element of $C_b(\mathbb{R})^*$, $T_a g$ is defined by the fact that for $f \in C_b(\mathbb{R})$,

$$\begin{aligned} \langle f, T_a g \rangle &= \int f(x)\,\overline{g(x-a)}\,dx \\ &= \int f(x+a)\,\overline{g(x)}\,dx \\ &= \langle T_{-a} f, g \rangle . \end{aligned}$$

Given an arbitrary $\mu \in C_b(\mathbb{R})^*$, we therefore define

$T_a\mu : C_b(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\langle f, T_a\mu \rangle = \langle T_{-a}f, \mu \rangle, \qquad f \in C_b(\mathbb{R}).$$

Exercise

Prove that $T_a\mu \in C_b(\mathbb{R})^*$ & $\|T_a\mu\| = \|\mu\|$.

Example

If $a \in \mathbb{R}$ then $\langle f, T_a\delta \rangle = \langle T_{-a}f, \delta \rangle = (T_{-a}f)(0) = f(a)$.

We often abuse notation & use a function notation for $T_a\delta$.

That is, we write "$T_a\delta(x) = \delta(x-a)$" and

$$\text{"} \int f(x)\, \delta(x-a)\, dx = f(a) \text{"}$$

We may justify this by saying "make a change of variables":

$$\text{"} \underbrace{\int f(x)\, \delta(x-a)\, dx}_{\langle f, T_a\delta \rangle} = \underbrace{\int f(x+a)\, \delta(x)\, dx}_{\langle T_{-a}f, \delta \rangle} = f(0+a) = f(a) \text{"}$$

but this is just shorthand for the definition $\langle f, T_a\delta \rangle = \langle T_{-a}f, \delta \rangle$.

Exercise

Extend the definition of translation, modulation, & dilation to $C_b(\mathbb{R})^*$ & $C_0(\mathbb{R})^*$.

Note: Later we will identify $C_0(\mathbb{R})^*$ with a space of measures.

Exercise

Abusing notation, let $2\delta(2x)$ be the dilation of δ by 2 that you defined in the preceding exercise. Show that

$$\delta(x) = 2\delta(2x)$$

Note: This is not an equality of functions but rather an equality of functionals on $C_b(\mathbb{R})$. In another notation, if we let $D_2\delta$ denote the dilation of δ by 2, then you are to prove that δ & $D_2\delta$ are the same functional on $C_b(\mathbb{R})$, i.e.,

$$\langle f, \delta \rangle = \langle f, D_2\delta \rangle \quad \forall f \in C_b(\mathbb{R}).$$

Definition

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in a Banach space X.

a. The finite linear span of $\{f_n\}$ is

$$\text{span}\{f_n\} = \left\{ \sum_{n=1}^{N} c_n f_n : N>0,\ c_1, \dots, c_N \in \mathbb{C} \right\}$$

b. The closed linear span of $\{f_n\}$ is the closure in X of the finite linear span, & it is denoted $\overline{\text{span}}\{f_n\}$.

c. $\{f_n\}$ is complete (or total or fundamental) in X if span$\{f_n\}$ is dense in X, i.e., $\overline{\text{span}}\{f_n\} = X$.

d. $\{f_n\}$ is minimal if $\forall m$, $f_m \notin \overline{\text{span}}\{f_n\}_{n \neq m}$.

Exercise Given $f_n \in X$, X a Banach space.

a. Show that $\{f_n\}$ is complete in X if & only if

$$T \in X^* \ \& \ \langle f_n, T \rangle = 0 \ \forall n \implies T = 0.$$

b. Show $\{f_n\}$ is minimal if & only if $\exists \tilde{f}_n \in X^*$ s.t. $\langle f_m, \tilde{f}_n \rangle = \delta_{mn} = \begin{cases} 1, & m=n \\ 0, & m \neq n. \end{cases}$

c. Show $\{f_n\}$ is mininimal & complete if & only if the sequence $\{\tilde{f}_n\}$ in part b is unique.

Hints: Continuity of T & Hahn-Banach.

Exercise

Fix $f \in L^2(\mathbb{R})$.
Prove that

$$\{T_a f\}_{a \in \mathbb{R}} \text{ is complete in } L^2(\mathbb{R}) \iff \hat{f}(\xi) \neq 0 \text{ a.e.}$$

Exercise

a. Prove that $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is not complete in $L^\infty[0,1]$.

b. Prove that $\{x e^{2\pi i n x}\}_{n \in \mathbb{Z} \setminus \{0\}}$ is complete in $L^2[0,1]$