## 3.3 Example of Distributions: Locally Integrable Functions

Definition

A measurable function $f: \mathbb{R} \to \mathbb{C}$ is locally integrable if

$$\forall \text{ compact } K, \quad \|f \cdot \chi_K\|_1 = \int_K |f(x)|\, dx < \infty.$$

Given $1 \le p \le \infty$, we define

$$L^p_{loc}(\mathbb{R}) = \{ f: \mathbb{R} \to \mathbb{C} \, : \, f \chi_K \in L^p(\mathbb{R}) \ \forall \text{ compact } K \subseteq \mathbb{R} \}.$$

Examples/Exercises

a. Every continuous function on $\mathbb{R}$ (bounded or unbounded) belongs to $L^1_{loc}(\mathbb{R})$, i.e., $C(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$.

b. $L^p(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$ for each $1 \le p \le \infty$.

c. $f(x) = 1/x \notin L^1_{loc}(\mathbb{R})$.

d. $L^p_{loc}(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$ for $1 \le p \le \infty$.

e. $BV_{loc}(\mathbb{R}) \subseteq L^\infty_{loc}(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$.

Remark

$L^1_{loc}(\mathbb{R})$ is a topological vector space whose topology is defined by the family of seminorms

$$\rho_K(f) = \|f \cdot \chi_K\|_1, \quad \text{compact } K \subseteq \mathbb{R}.$$

WLOG, we can use a countable collection of seminorms

$$\rho_n(f) = \| f \cdot \chi_{[-n,n]} \|_1, \quad n \in \mathbb{N}.$$

This topology is Hausdorff, i.e.,

$$\rho_n(f) = 0 \ \forall n \in \mathbb{N} \implies f = 0 \text{ a.e.}$$

Convergence in this topology is defined as follows.

If $f_k, f \in L^1_{loc}(\mathbb{R})$, then $f_k \to f$ in $L^1_{loc}(\mathbb{R})$

means:

$$\forall n \in \mathbb{N}, \quad \rho_n(f - f_k) = \| (f - f_k) \chi_{[-n,n]} \|_1 \to 0$$

as $k \to \infty$.

See the appendix for more detailed discussion of topologies defined by families of seminorms. In particular, it is the fact that the topology on $L^1_{loc}(\mathbb{R})$ is determined by a countable collection of seminorms that allows us to define convergence using ordinary sequences instead of nets. Indeed,

$$\rho(f,g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f-g)}{1 + \rho_n(f-g)}$$

defines a metric on $L^1_{loc}(\mathbb{R})$ that determines the same topology. Further, $L^1_{loc}(\mathbb{R})$ is complete w.r.t. this metric, and therefore is an example of a Fréchet space.

Exercise

a. Show that

$$f \in L^1_{loc}(\mathbb{R}),\ g \in C_c^m(\mathbb{R}) \implies f * g \in C^m(\mathbb{R}).$$

Hint:

$$(f*g)(x) = \int f(y)\,g(x-y)\,dy = \int_{K_x} f(y)\,g(x-y)\,dy$$

where $K_x = x - \mathrm{supp}(g)$ is compact.

b. Give an example of $f \in L^1_{loc}(\mathbb{R})$, $g \in C_c(\mathbb{R})$ such that $f * g$ is unbounded.

c. Let $k \in C_c(\mathbb{R})$ with $\int k(x)\,dx = 1$ be given, and set $k_\lambda(x) = \lambda k(\lambda x)$. Prove that if $f \in L^1_{loc}(\mathbb{R})$ then $f * k_\lambda \to f$ in the topology of $L^1_{loc}(\mathbb{R})$, i.e.,

$$\forall \text{ compact } K \subseteq \mathbb{R}, \quad \| (f - f*k_\lambda)\,\chi_K \|_1 \to 0 \quad \text{as } \lambda \to \infty.$$

Hint: Imitate the proof of $f * k_\lambda \to f$ in $L^1(\mathbb{R})$ when $f \in L^1(\mathbb{R})$.

Additional hint. Take $K = [-R, R]$. If $\mathrm{supp}(k) \subseteq [-1,1]$ then $\mathrm{supp}(k_\lambda) \subseteq [-\frac{1}{\lambda}, \frac{1}{\lambda}]$.

Alternative (better): We know $f\chi_K * k_\lambda \to f\chi_K$ in $L^1$-norm. So, show

$$\| (f\chi_K) * k_\lambda - (f*k_\lambda)\,\chi_K \|_1 \to 0.$$

Theorem

$L^1_{loc}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$.

More precisely, if $g \in L^1_{loc}(\mathbb{R})$, then

$$\langle f, L_g \rangle = \int f(x)\, \overline{g(x)}\, dx, \quad f \in C_c(\mathbb{R}),$$

defines a distribution $L_g \in \mathcal{D}'(\mathbb{R})$, and the mapping $g \longmapsto L_g$ is a continuous injection of $L^1_{loc}(\mathbb{R})$ into $\mathcal{D}'(\mathbb{R})$.

Proof:

Fix $g \in L^1_{loc}(\mathbb{R})$. First we will show that $L_g$ is a distribution. Suppose that $f_n \to 0$ in $C_c^\infty(\mathbb{R})$. Then $\exists$ compact $K \subseteq \mathbb{R}$ s.t. $\mathrm{supp}(f_n) \subseteq K$ $\forall n$, & further $\|f_n^{(k)}\|_\infty \to 0$ for each $k \geq 0$. Hence

$$\begin{aligned} |\langle f_n, L_g \rangle| &\leq \int_K |f(x)|\, |g(x)|\, dx \\ &\leq \|f_n\|_\infty \, \|g \cdot \chi_K\|_1 \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $g \in \mathcal{D}'(\mathbb{R})$.

Since $\mathcal{U}: g \mapsto L_g$ is antilinear, to show it is injective we just have to show $\ker(\mathcal{U}) = \{0\}$. So, suppose $L_g = 0$. Let $k \in C_c^\infty(\mathbb{R})$ with $\int k(x)\,dx = 1$ be fixed, & set $k_\lambda = \lambda k(\lambda x)$. Then by the preceding exercise, $g * k_\lambda \to g$ in $L^1_{loc}(\mathbb{R})$, i.e., for any compact $K \subseteq \mathbb{R}$, $(g*k_\lambda)\chi_K \to g\chi_K$ in $L^1(\mathbb{R})$. But if we set $\tilde{k}_\lambda(x) = \overline{k_\lambda(-x)}$, then

$$
\begin{aligned}
(g*k_\lambda)(x) &= \int g(y)\, k_\lambda(x-y)\,dy \\
&= \int g(y)\, \overline{\tilde{k}_\lambda(y-x)}\,dy \\
&= \overline{\langle T_x \tilde{k}_\lambda, L_g \rangle} \\
&= 0.
\end{aligned}
$$

Hence $g\chi_K = 0$ ae for each compact $K$, so $g = 0$ a.e.

Finally, to show $\mathcal{U}$ is continuous, suppose that $g_n, g \in L^1_{loc}(\mathbb{R})$ & $g_n \to g$ in $L^1_{loc}(\mathbb{R})$.

Given $f \in C_c^\infty(\mathbb{R})$, let $K = \text{supp}(f)$. Then

$$|\langle f, L_g\rangle - \langle f, L_{g_n}\rangle|$$

$$\leq \int_K |f(x)|\, |g(x) - g_n(x)|\, dx$$

$$\leq \|f\|_\infty \, \|(g-g_n)\chi_K\|_1$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

Hence $L_{g_n} \longrightarrow L_g$ in $\mathcal{D}'(\mathbb{R})$.

Thus $\mathcal{U}$ is continuous. $\blacksquare$

<u>Exercise</u>

Suppose that $g \in L^1_{loc}(\mathbb{R}) \setminus \{0\}$. Show that

$\exists f \in C_c^\infty(\mathbb{R})$ s.t. $\langle f, g\rangle \neq 0$.

Exercise

a. Show that
$$\delta : C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$$
$$f \longmapsto f(0)$$

defines a distribution, i.e., $\delta \in \mathcal{D}'(\mathbb{R})$.

b. Show that $\delta \notin L^1_{loc}(\mathbb{R})$, i.e., $\nexists\, g \in L^1_{loc}(\mathbb{R})$ s.t. $\delta = L_g$.

c. Let $\{k_\lambda\}_{\lambda>0}$ be any approximate identity. Prove that
$$k_\lambda \longrightarrow \delta \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ as } \lambda \to \infty.$$

That is,
$$\forall f \in C_c^\infty(\mathbb{R}), \qquad \langle f, k_\lambda \rangle \longrightarrow \langle f, \delta \rangle = f(0).$$

Hint: Write $\langle f, k_\lambda \rangle$ as a convolution evaluated at $x=0$.

Remark: The function $1/x$ does not belong to $\mathcal{D}'(\mathbb{R})$; indeed,

$$\langle f, \tfrac{1}{x} \rangle = \int \frac{f(x)}{x}\, dx \qquad (*)$$

is not even necessarily defined for all $f \in C_c^\infty(\mathbb{R})$.

However, by appropriately interpreting the integral, we can create a distribution based on $1/x$.

Definition

The principle value of $1/x$, is $pv(\frac{1}{x}) : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ given by

$$\langle \varphi, pv(\tfrac{1}{x}) \rangle = \lim_{T \to \infty} \int_{\frac{1}{T} < |x| < T} \frac{\varphi(x)}{x}\, dx.$$

Remark

This is not just an intrepretation of $(*)$ as an improper integral — that would be the case if we defined the integral to be

$$\lim_{\substack{a \to 0^+, b \to \infty, \\ c \to -\infty, d \to 0^-}} \int_a^b \frac{\varphi(x)}{x}\, dx + \int_c^d \frac{\varphi(x)}{x}\, dx$$

with $a, b, c, d$ converging independently. Instead, we integrate only on symmetric intervals.

Exercises

a. Show that $\langle \varphi, pv(\frac{1}{x}) \rangle$ is well-defined for each $\varphi \in C_c^\infty(\mathbb{R})$.

Hints: Split the integral as $\int_{\frac{1}{T}<|x|<1}$ & $\int_{1<|x|<T}$.

Recall $\varphi$ is differentiable; in particular $\frac{\varphi(x)-\varphi(0)}{x-0}$ converges as $x \to 0$.

b. Show that $pv(\frac{1}{x}) \in \mathcal{D}'(\mathbb{R})$.*

c. Recall an earlier exercise that showed that $\delta(x) = 2\delta(2x)$ (in the sense of distributions). Show that

$$pv\left(\frac{1}{x}\right) = 2\, pv\left(\frac{1}{2x}\right)$$

again in the sense of distributions. Thus the functional equation

$$f(x) = 2f(2x)$$

does not have a unique distributional solution.

Hints for b: Split the integral into $\int_{1<|x|<T}$ & $\int_{\frac{1}{T}<|x|<1}$.

For the second, by the MVT,

$$\left|\frac{\varphi(x)-\varphi(0)}{x-0}\right| = |\varphi'(c)| \leq \|\varphi'\|_\infty$$

Exercise: Not every linear functional on $C_c^\infty(\mathbb{R})$ is continuous.

a. Fix any nonzero $\varphi \in C_c^\infty(\mathbb{R})$. Show that

$$\frac{1}{n}\varphi \to 0 \quad \text{and} \quad \frac{1}{n} T_{\frac{1}{n}}\varphi \to 0 \quad \text{in } C_c^\infty(\mathbb{R}).$$

b. By an earlier exercise, $\{\frac{1}{n} T_{\frac{1}{n}}\varphi\}_{n\in\mathbb{N}}$ is finitely linearly independent. By Zorn's Lemma* (which is equivalent to the Axiom of Choice), $\exists$ Hamel basis (= vector space basis) for $C_c^\infty(\mathbb{R})$ containing $\{\frac{1}{n} T_{\frac{1}{n}}\varphi\}_{n\in\mathbb{N}}$, say $\{\psi_\alpha\}_{\alpha\in I}$. Define $\mu : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ as follows.

If $f = 0$, set $\langle f, \mu\rangle = 0$. Define

$$\langle \psi_\alpha, \mu\rangle = \begin{cases} 1, & \text{if } \psi_\alpha = \frac{1}{n} T_{\frac{1}{n}}\varphi \text{ for some } n \\ 0, & \text{otherwise.} \end{cases}$$

If $f \in C_c^\infty(\mathbb{R})$, $f \neq 0$, then by definition of Hamel basis $\exists$ unique $\alpha_1, \dots, \alpha_N \in I$ & nonzero $c_1, \dots, c_N \in \mathbb{C}$ s.t.

$$f = \sum_{k=1}^{N} c_k \psi_{\alpha_k}.$$

---

*What is yellow and equivalent to the Axiom of Choice? Zorn's Lemon!

Define

$$\langle f, \mu \rangle = \sum_{k=1}^{N} c_k \langle \psi_{\alpha_k}, \mu \rangle.$$

Prove that $\mu$ is linear.

c. Prove that $\mu$ is not continuous.

<u>Remark/Exercise</u>

A simpler version of this idea shows that if $X$ is any $\infty$-dimensional normed space, then $\exists$ unbounded linear functionals on $X$.

Let $\{\psi_\alpha\}_{\alpha \in I}$ be any Hamel basis for $X$. Choose a countable subset $\{\psi_n\}_{n \in \mathbb{N}}$. WLOG, we may assume $\|\psi_\alpha\| = 1$ $\forall \alpha$ (divide $\psi_\alpha$ by its length if necessary).

Define

$$\langle \psi_\alpha, \mu \rangle = \begin{cases} n, & \text{if } \psi_\alpha = \psi_n \text{ for some } n \\ 0, & \text{otherwise.} \end{cases}$$

Extend $\mu$ linearly to $X$. Show $\mu$ is unbounded.