3.4 Some "generalized function" properties of distributions: support ~~and differentiation~~.

The space $\mathcal{D}'(\mathbb{R})$ is very large. Yet distributions have many "generalized function" properties. For example, we can define the support of a distribution.

Example

If $U \subseteq \mathbb{R}\setminus\{0\}$ is open & $\operatorname{supp}(f) \subseteq U$, then

$$\langle f, \delta \rangle = f(0) = 0.$$

We can only have $\langle f, \delta\rangle \neq 0$ if $0 \in \operatorname{supp}(f)$.

Definition

Let $T \in \mathcal{D}'(\mathbb{R})$ be given.

a. We say T is zero on an open set $U \subseteq \mathbb{R}$ if

$$f \in C_c^\infty(\mathbb{R}),\ \operatorname{supp}(f) \subseteq U \quad \Rightarrow \quad \langle f, T\rangle = 0.$$

b. The support of T is

$$\operatorname{supp}(T) = \bigcap \{ F \subseteq \mathbb{R} : F \text{ closed \& } T \text{ is zero on } \mathbb{R}\setminus F\}.$$

Thus $\operatorname{supp}(T)$ is the smallest closed set such that T is zero on $\mathbb{R}\setminus\operatorname{supp}(T)$.

Exercise

Let $\mu \in \mathcal{D}'(\mathbb{R})$ have compact support, i.e., $\mu \in \mathcal{E}'(\mathbb{R})$.

Let $U \supseteq \text{supp}(\mu)$ be open, & suppose $\theta \in C_c^\infty(\mathbb{R})$

is s.t. $\theta = 1$ on U. Show that $\theta\mu = \mu$, i.e.,

$$\forall f \in C_c^\infty(\mathbb{R}), \quad \langle f, \mu \rangle = \langle f, \theta\mu \rangle = \langle f\theta, \mu \rangle.$$

Exercise

Given $\mu \in \mathcal{D}'(\mathbb{R})$ & $\theta \in C^\infty(\mathbb{R})$, show that

$$\text{supp}(\theta\mu) \subseteq \text{supp}(\theta) \cap \text{supp}(\mu).$$

Recall that we defined translation, modulation, & dilation of functionals "by duality". Thus if $\mu \in \mathcal{D}'(\mathbb{R})$ & $a \in \mathbb{R}$ then $T_a\mu$ is given by

$$\langle f, T_a\mu \rangle = \langle T_{-a}f, \mu \rangle, \qquad f \in C_c^\infty(\mathbb{R}).$$

Exercises

a. $\operatorname{supp}(\delta) = \{0\}$.

b. If $g \in L^1_{loc}(\mathbb{R})$ then $\operatorname{supp}(g) = \operatorname{supp}(T_g)$, where $T_g \in \mathcal{D}'(\mathbb{R})$ is defined by

$$\langle f, T_g \rangle = \int f(x)\, \overline{g(x)}\, dx, \qquad f \in C_c^\infty(\mathbb{R}).$$

c. If $\mu \in \mathcal{D}'(\mathbb{R})$ & $a \in \mathbb{R}$ then $T_a\mu \in \mathcal{D}'(\mathbb{R})$, & $\operatorname{supp}(T_a\mu) = \operatorname{supp}(\mu) + a$.

Exercise

Given $\mu \in \mathcal{D}'(\mathbb{R})$ & $f \in C_c^\infty(\mathbb{R})$, show

$$\operatorname{supp}(f * \mu) \subseteq \overline{\operatorname{supp}(f) + \operatorname{supp}(\mu)} \qquad \left(\begin{array}{c}\text{closure not needed} \\ \text{if } \operatorname{supp}(\mu) \text{ is} \\ \text{compact}\end{array}\right)$$

Definition

Given $f_n, g \in C^\infty(\mathbb{R})$, we say that $f_n \longrightarrow g$ in $C^\infty(\mathbb{R})$ if

$$\forall \text{compact } K \subseteq \mathbb{R}, \quad \forall k \geq 0,$$

$$\lim_{n\to\infty} \| (g^{(k)} - f_n^{(k)}) \cdot \chi_K \|_\infty = 0.$$

Remark

In other words, the topology on $C^\infty(\mathbb{R})$ is generated by the family of seminorms

$$\rho_{k,K}(f) = \| f^{(k)} \cdot \chi_K \|_\infty,$$

where $k \in \mathbb{N}$ & $K \subseteq \mathbb{R}$ is compact. WLOG, by considering $K = [-n, n]$ we may reduce to a countable family of seminorms.

Definition

$$\mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^* = \{\mu : C^\infty(\mathbb{R}) \to \mathbb{C} : \mu \text{ is linear \& continuous}\}$$

Exercise

Show that $C_c^\infty(\mathbb{R})$ is dense in $C^\infty(\mathbb{R})$ (in the topology of $C^\infty(\mathbb{R})$). That is, show that if $f \in C^\infty(\mathbb{R})$ then $\exists\, f_n \in C_c^\infty(\mathbb{R})$ s.t. $f_n \longrightarrow f$ in $C^\infty(\mathbb{R})$.

__Exercise__

Suppose that $\mu \in \mathcal{D}'(\mathbb{R})$ & $\mathrm{supp}(\mu)$ is compact. By Urysohn's Lemma, $\exists$ a smooth cutoff function $\theta \in C_c^\infty(\mathbb{R})$ s.t.

$$0 \le \theta \le 1 \quad \& \quad \theta = 1 \text{ on } \mathrm{supp}(\mu).$$

Given $f \in C^\infty(\mathbb{R})$, define

$$\langle f, \mu \rangle = \langle f\theta, \mu \rangle.$$

Show that $\mu : C^\infty(\mathbb{R}) \to \mathbb{C}$ defined in this way is a continuous linear functional, & its definition is independent of the choice of cutoff function.

& further, if $f \in C_c^\infty(\mathbb{R})$ then this definition of $\langle f, \mu \rangle$ agrees with the original definition.

Thus every compactly supported distribution extends naturally to an element of $\mathcal{E}'(\mathbb{R})$.

In fact, the next result shows that the converse is also true

Theorem

$\mathcal{E}'(\mathbb{R})$ is the space of compactly supported distributions, i.e.,

$$\mathcal{E}'(\mathbb{R}) = \{\mu \in \mathcal{D}'(\mathbb{R}) : \operatorname{supp}(\mu) \text{ is compact}\}.$$

Proof:

The preceding exercise showed that every compactly supported distribution ~~is compact~~ belongs to $\mathcal{E}'(\mathbb{R})$.

So, suppose that $\mu \in \mathcal{E}'(\mathbb{R})$. First we will show that $\mu \in \mathcal{D}'(\mathbb{R})$, or more precisely, its restriction to $C_c^\infty(\mathbb{R})$ belongs to $\mathcal{D}'(\mathbb{R})$.

Suppose that $f_n \in C_c^\infty(\mathbb{R})$ & $f_n \to 0$ in $C_c^\infty(\mathbb{R})$. Then $\exists$ compact $K \subseteq \mathbb{R}$ s.t. $\operatorname{supp}(f_n) \subseteq K$ $\forall n$, and

$$\forall k \in \mathbb{N}, \quad \lim_{n\to\infty} \| f_n^{(k)} \|_\infty = 0.$$

Hence if K' is any compact set, then, since $f_n^{(k)} = 0$ off K,

$$\| f_n^{(k)} \cdot \chi_{K'} \|_\infty = \| f_n^{(k)} \cdot \chi_{K \cap K'} \|_\infty$$
$$\leq \| f_n^{(k)} \|_\infty$$
$$\longrightarrow 0 \quad \text{as } n \to \infty.$$

Thus $f_n \to 0$ in $C^\infty(\mathbb{R})$. Hence

$\langle f_n, \mu \rangle \longrightarrow 0$, so μ is continuous on $C_c^\infty(\mathbb{R})$.

Now, by a theorem in the appendix, since the topology on $C^\infty(\mathbb{R})$ is given by the family of seminorms

$$\rho_{k,K}(f) = \| f^{(k)} \cdot \chi_K \|_\infty, \quad k \in \mathbb{N}, \ K \text{ compact},$$

since μ is continuous on $C^\infty(\mathbb{R})$ $\exists N$, $\exists C > 0$, $\exists K = [-m, m]$ s.t.

(*) $\forall f \in C^\infty(\mathbb{R})$, $\displaystyle |\langle f, \mu \rangle| \leq C \sum_{k=0}^{N} \| f^{(k)} \cdot \chi_K \|_\infty$.

Hence if $f \in C_c^\infty(\mathbb{R})$ & $\mathrm{supp}(f) \subseteq \mathbb{R} \setminus K$, then $\langle f, \mu \rangle = 0$. Therefore $\mathrm{supp}(\mu) \subseteq K$, i.e., μ has compact support. ∎

Exercise

Let $\mu \in \mathcal{E}'(\mathbb{R}) = C^\infty(\mathbb{R})^*$ be given.

a. Show that if $f \in C^\infty(\mathbb{R})$ & $\operatorname{supp}(f) \subseteq \mathbb{R} \setminus \operatorname{supp}(\mu)$

then $\langle f, \mu \rangle = 0$.

Hint: $C_c^\infty(\mathbb{R})$ is dense in $C^\infty(\mathbb{R})$, & μ is continuous.

b. Prove that if $\theta \in C^\infty(\mathbb{R})$ then $\theta\mu \in \mathcal{E}'(\mathbb{R})$,

where $\langle f, \theta\mu \rangle = \langle f\bar{\theta}, \mu \rangle$ for $f \in C^\infty(\mathbb{R})$.

c. Prove that if $\theta \in C^\infty(\mathbb{R})$ & $\theta = 1$ on an open

set $U \supseteq \operatorname{supp}(\mu)$, then $\theta\mu = \mu$.

As an application, we show that all compactly supported distributions have finite order.

Theorem

Suppose that $\mu \in \mathcal{D}'(\mathbb{R})$ has compact support, i.e., $\mu \in \mathcal{E}'(\mathbb{R})$. Then μ has finite order. In fact, $\exists N \geq 0$, $\exists C > 0$ s.t.

$$\forall f \in C_c^\infty(\mathbb{R}), \qquad |\langle f, \mu \rangle| \leq C \max\{\|f^{(n)}\|_\infty\}_{n=0}^N .$$

Proof:

Let $K = \operatorname{supp}(\mu)$, & let U be a bounded open set containing K. Then by Urysohn's Lemma (proved in Chapter 1), $\exists \theta \in C_c^\infty(\mathbb{R})$ s.t. $\theta = 1$ on U.

An earlier exercise shows that $\theta\mu = \mu$, which means that

$$\forall f \in C_c^\infty(\mathbb{R}), \qquad \langle f\theta, \mu \rangle = \langle f, \mu \rangle .$$

By a previous theorem, $\exists N \geq 0$, $\exists C > 0$ s.t.

$$f \in C_c^\infty(\mathbb{R}),\ \operatorname{supp}(f) \subseteq \overset{\operatorname{supp}(\theta)}{\cancel{K}} \Rightarrow |\langle f, \mu \rangle| \leq C \max\{\|f^{(n)}\|_\infty\}_{n=0}^N .$$

We wish to extend this to all $f \in C_c^\infty(\mathbb{R})$. By the Leibnitz rule for differentiation, we have

$$(f\bar{\theta})^{(n)}(x) = \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x)\, \theta^{(n-j)}(x).$$

Hence if $f \in C_c^\infty(\mathbb{R})$, then

$$
\begin{aligned}
|\langle f,\mu\rangle| &= |\langle f\bar{\theta},\mu\rangle| \\
&\le C \max\left\{ \|(f\bar{\theta})^{(n)}\|_\infty \right\}_{n=0}^{N} \\
&\le C \max_{n=0,\dots,N} \sum_{j=0}^{N} \binom{n}{j} \|f^{(j)}\|_\infty \|\theta^{(n-j)}\|_\infty \\
&\le C' \max\left\{ \|f^{(j)}\|_\infty \right\}_{j=0}^{N}. \quad \blacksquare
\end{aligned}
$$

Next we show that we can, ~~we~~ at least sometimes, define the convolution of two distributions. Recall from an earlier theorem that if $f, g \in C_c^\infty(\mathbb{R})$ & $\mu \in \mathcal{D}'(\mathbb{R})$ then

$$\langle f, g * \mu \rangle = \langle f * \tilde{g}, \mu \rangle \qquad (*)$$

Further,

$$\begin{aligned} \langle f, \tilde{g} \rangle &= \int f(x) \overline{\tilde{g}(x)}\, dx \\ &= \int f(x)\, g(-x)\, dx \\ &= \int f(-x)\, g(x)\, dx \\ &= \overline{\langle \tilde{f}, g \rangle}. \end{aligned}$$

<u>Exercise</u>

Given $\mu \in \mathcal{D}'(\mathbb{R})$, show that $\tilde{\mu} \in \mathcal{D}'(\mathbb{R})$, where

$$\langle f, \tilde{\mu} \rangle = \overline{\langle \tilde{f}, \mu \rangle}, \qquad f \in C_c^\infty(\mathbb{R}).$$

Show that if μ has compact support then so does $\tilde{\mu}$.

Note that $\langle \tilde{f}, \tilde{\mu} \rangle = \overline{\langle f, \mu \rangle}$

Hence, given $\mu, \nu \in \mathcal{D}'(\mathbb{R})$, we can attempt to defining $\mu * \nu$ by extending $(*)$, i.e., declaring

$$(**) \quad \langle f, \mu * \nu \rangle = \langle f * \tilde{\mu}, \nu \rangle, \quad f \in C_c^\infty(\mathbb{R}).$$

Unfortunately, in general we will only have $f * \tilde{\mu} \in C^\infty(\mathbb{R})$, not $C_c^\infty(\mathbb{R})$, so $(**)$ need not make sense. But if impose that μ have compact support, then $f * \tilde{\mu} \in C_c^\infty(\mathbb{R})$, so $(**)$ makes sense. Likewise, if μ is arbitrary but ν has compact support then $(**)$ is again well-defined.

__Definition__
If $\mu \in \mathcal{E}'(\mathbb{R})$, $\nu \in \mathcal{D}'(\mathbb{R})$ or $\mu \in \mathcal{D}'(\mathbb{R})$, $\nu \in \mathcal{E}'(\mathbb{R})$, then $\mu * \nu : C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{C}$ is defined by $(**)$.

__Exercise__
Prove that in either case, $\mu * \nu \in \mathcal{D}'(\mathbb{R})$
Hint: an earlier theorem showed that

$f \longmapsto f * \mu$ is a continuous linear mapping of $C_c^\infty(\mathbb{R})$ into $C^\infty(\mathbb{R})$.

<u>Exercise</u>

Let $\mu, \nu \in \mathcal{D}'(\mathbb{R})$, with at least one having compact support. Prove that

$$\forall f \in C_c^\infty(\mathbb{R}), \quad f * (\mu * \nu) = (f * \mu) * \nu$$

Hint: Show that $(\tilde{f} * \tilde{\mu})^\sim = f * \mu$.

<u>Exercise</u> (see Rudin, F.A., 2^{nd} ed, p. 180)

Let $H = \chi_{[0,\infty)}$ be the Heaviside function, & let $\underline{1}$ be the constant function. Compute the following directly (& justify their existence):

$$H * \delta' \qquad \delta' * H$$

$$\delta' * \underline{1} \qquad \underline{1} * \delta'$$

$$(H * \delta') * \underline{1} \qquad \underline{1} * (\delta' * H)$$

$$H * (\delta' * \underline{1}) \qquad (\underline{1} * \delta') * H$$

Prove that $(H * \delta') * \underline{1} \neq H * (\delta' * \underline{1})$,

$(\underline{1} * \delta') * H \neq \underline{1} * (\delta' * H)$.

Why does this not contradict the preceding exercise?

Exercise (See Benedetto)

Let $\mu = \sum_{n=0}^{\infty}{}' (-1)^n \, \delta_n * D^n \delta$ (where $\delta_n = T_n \delta$).

Show that $\langle f, \mu \rangle = \sum_{n=0}^{\infty}{}' f^{(n)}(n)$, & relate this to an earlier exercise.

Remark

This is not the end of the story on convolution of distributions; however, it is as far as we shall pursue the topic. See Benedetto

Horvath

others