3.6 Tempered Distribution

Since $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$, we know that the F.T. maps $C_c^\infty(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. However, by the Paley-Wiener theorem, if f has compact support then $\hat{f}$ cannot be compactly supported. Hence the F.T. does not map $C_c^\infty(\mathbb{R})$ into itself. A consequence of this is that, unlike such operators as translation, modulation, dilation, differentiation, we ~~can~~ cannot extend the F.T. to all of $C_c^\infty(\mathbb{R})^* = \mathcal{D}'(\mathbb{R})$. Instead, we must restrict the extension to a space properly contained in $\mathcal{D}'(\mathbb{R})$, namely $\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^*$, the space of tempered distributions.

To emphasize this point, if we extend the def. of the F.T., then this definition should extend the existing functional properties of the F.T. In particular, if $f, g \in L^2(\mathbb{R})$ then we have the Parseval equality:

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$$

Restricting to $f \in \mathcal{S}(\mathbb{R})$, since the F.T. maps $\mathcal{S}(\mathbb{R})$ onto itself, this implies that if $g \in L^2(\mathbb{R})$ then

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \hat{g} \rangle = \langle \check{f}, g \rangle.$$

This property tells us how to extend the F.T. to elements in $\mathcal{S}'(\mathbb{R})$: if $\mu \in \mathcal{S}'(\mathbb{R})$ then $\hat{\mu}$ is the element of $\mathcal{S}'(\mathbb{R})$ defined by

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \hat{\mu} \rangle = \langle \check{f}, \mu \rangle.$$

Of course, we must demonstrate that this does define an element of $\mathcal{S}'(\mathbb{R})$, & to do that we must first review the meaning of continuity of a linear functional $\mu : \mathcal{S}(\mathbb{R}) \longrightarrow \mathbb{C}$, or, in equivalent terms, the definition of the topology on $\mathcal{S}(\mathbb{R})$.

<u>Definition</u>

Given $f_k, g \in \mathcal{S}(\mathbb{R})$, we say that $f_k \to g$ in $\mathcal{S}(\mathbb{R})$ if

$$\forall m, n \geq 0, \qquad \| x^m g^{(n)}(x) - x^m f_k^{(n)}(x) \|_\infty \to 0 \quad \text{as } k \to \infty.$$

<u>Definition</u>

a. A linear functional $\mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is <u>continuous</u> if

$$f_k \to 0 \text{ in } \mathcal{S}(\mathbb{R}) \implies \langle f_k, \mu \rangle \to 0.$$

b. The space of <u>tempered distributions</u> is

$$\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^* = \{ \mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C} : \mu \text{ is a continuous linear functional} \}$$

<u>Exercise</u>

Prove that $\delta \in \mathcal{S}'(\mathbb{R})$.

<u>Exercise</u>

Show that if $f_k, f \in \mathcal{S}(\mathbb{R})$ & $f_k \to f$ in $\mathcal{S}(\mathbb{R})$, then $f_k' \to f'$ in $\mathcal{S}(\mathbb{R})$.

Example

Recall that $L^1_{loc}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$. In particular,

$g(x) = e^{x^2} \in L^1_{loc}(\mathbb{R}) \subseteq \mathcal{D}'(\mathbb{R})$. However,

if we take $f(x) = e^{-x^2} \in \mathcal{S}(\mathbb{R})$, then

$$\langle f, g \rangle = \int e^{-x^2} e^{x^2}\, dx$$

does not even converge. Hence g does not define an element of $\mathcal{S}'(\mathbb{R})$.

Exercise

Prove that if $g \in L^1_{loc}(\mathbb{R})$ &

$\exists$ polynomial p s.t.

$$|g(x)| \leq |p(x)| \quad \forall\, x \text{ large enough},$$

then $g \in \mathcal{S}'(\mathbb{R})$.

Thus $\mathcal{S}'(\mathbb{R})$ includes all locally integrable functions with polynomial growth at ∞. However, the requirement of polynomial growth is only sufficient, not necessary, as the following example shows.

<u>Example</u>

We have $g(x) = \sin e^x \in L^\infty(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$. Since $f \in C^\infty(\mathbb{R})$, its distributional derivative coincides with its pointwise derivative, so we know that

$$g'(x) = e^x \cos e^x \in \mathcal{D}'(\mathbb{R}).$$

There does not exist any polynomial p that dominates g'.

Yet if $f \in \mathcal{S}(\mathbb{R})$ then

$$\lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b f(x)\, e^x \cos e^x \, dx$$

$$= \lim_{\substack{a \to -\infty \\ b \to \infty}} \left[f(b) \sin b - f(a) \sin a - \int_a^b f'(x) \sin e^x \, dx \right]$$

$$= -\int_{-\infty}^{\infty} f'(x) \sin e^x \, dx$$

$$= -\langle f', g \rangle.$$

Hence $\langle f, g' \rangle = \int_{-\infty}^{\infty} f(x)\, e^x \cos e^x \, dx = -\langle f', g \rangle$ well-defined. Further, if $f_k \to f$ in $\mathcal{S}(\mathbb{R})$, then since $f_k' \to f$ in $\mathcal{S}(\mathbb{R})$, we have that

$$\langle f_k, g' \rangle = -\langle f_k', g \rangle \to -\langle f', g \rangle = \langle f, g' \rangle.$$

Hence $g' \in \mathcal{S}(\mathbb{R})$.

<u>Remark</u> (See Folland, Real Analysis, 2nd ed, p. 294)

Although f' is not bounded by a polynomial, "on average" it is not "too large", because of its rapid oscillations.

<u>Exercise</u>

$e^{+|x|} \notin \mathcal{S}'(\mathbb{R})$.

Exercise

Show that for each $1 \leq p \leq \infty$, $L^p(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$.

Note that if $f \in L^p(\mathbb{R})$ with $p < \infty$, then f need not have polynomial growth at ∞.

Remark

$\mathcal{S}(\mathbb{R})$ is a topological vector space whose topology is defined by the countably family of seminorms

$$\rho_{mn}(f) = \|x^m f^{(n)}(x)\|_\infty, \qquad m,n \geqslant 0.$$

If we define "balls"

$$B_\varepsilon^{mn}(f) = \{ g \in \mathcal{S}(\mathbb{R}) : \rho_{mn}(f-g) < \varepsilon \},$$

then the topology on $\mathcal{S}(\mathbb{R})$ is generated by these "balls".

More precisely, the collection of of all finite intersections of the $B_\varepsilon^{mn}(f)$ is a base for the topology, meaning every open set is a union of base elements. Consequently, if $U \subseteq \mathcal{S}(\mathbb{R})$ is open & $f \in U$, then $\exists$ finitely many m_j, n_j, ε_j s.t.

$$\bigcap_{j=1}^{N} B_{m_j,n_j}^{\varepsilon_j}(f) \subseteq U.$$

Taking as a definition that $\mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ is continuous if $\mu^{-1}(V)$ is open in $\mathcal{S}(\mathbb{R})$ for each open $V \subseteq \mathbb{C}$, it is shown in the Appendix that our definition of continuity is a consequence.

Because of the following two facts:

- there are countably many seminorms,
- the topology is Hausdorff, which is equivalent to

$$\rho_{mn}(f) = 0 \quad \forall m, n \geq 0 \quad \Rightarrow \quad f = 0,$$

we can create a <u>metric</u> on $\mathcal{S}(\mathbb{R})$ that generates the same topology, namely,

$$d(f,g) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \frac{\rho_{mn}(f-g)}{1 + \rho_{mn}(f-g)}.$$

That is, $f_k \to f$ in $\mathcal{S}(\mathbb{R})$ if & only if $d(f_k, f) \to 0$.

Consequently, the notion of Cauchy sequences in $\mathcal{S}(\mathbb{R})$ makes sense, & we can ask if $\mathcal{S}(\mathbb{R})$ is <u>complete</u>, i.e., is every Cauchy sequence convergent?

In fact it is (exercise), so $\mathcal{S}(\mathbb{R})$ is a vector space that is a complete metric space. Such a space is called <u>a Fréchet space</u>. Unfortunately, the metric on $\mathcal{S}(\mathbb{R})$ is not induced by any norm, so $\mathcal{S}(\mathbb{R})$ is not a Banach space.

Expanded discussion of these issues is given in the Appendix.

Remark

Because the topology on $\mathcal{S}(\mathbb{R})$ is given by a family of seminorms, there is an analogue of "continuity = boundedness." The following is a consequence of a result proved in the Appendix.

Theorem

Let $\mu : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ be a linear functional. Then TFAE:

a. μ is continuous, i.e., $\mu \in \mathcal{S}'(\mathbb{R})$,

b. $\exists M, N \geq 0$, $\exists C > 0$ s.t.

$$\forall f \in \mathcal{S}(\mathbb{R}), \qquad |\langle f, \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \|x^m f^{(n)}(x)\|_\infty$$

Definition

The topology on $\mathcal{S}'(\mathbb{R})$ is the weak* topology, i.e.,

if $\mu_n, \mu \in \mathcal{S}'(\mathbb{R})$, then $\mu_n \longrightarrow \mu$ in $\mathcal{S}'(\mathbb{R})$ means

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \mu_n \rangle \longrightarrow \langle f, \mu \rangle.$$

Exercise

Suppose that $\mu_n \longrightarrow \mu$ in $\mathcal{S}'(\mathbb{R})$, & $\exists$ compact $K \subseteq \mathbb{R}$ s.t. $\operatorname{supp}(\mu_n) \subseteq K$ $\forall n$. Show that $\operatorname{supp}(\mu) \subseteq K$.

Exercise

Suppose that $f_k, f \in \mathcal{S}(\mathbb{R})$ & $f_k \longrightarrow f$ in $\mathcal{S}(\mathbb{R})$.

a. Prove that if $g \in \mathcal{S}(\mathbb{R})$ then $f_k g \longrightarrow fg$ in $\mathcal{S}(\mathbb{R})$.

b. Prove that if $g \in \mathcal{S}(\mathbb{R})$ then $f_k * g \longrightarrow f*g$ in $\mathcal{S}(\mathbb{R})$.

Hint: $(f*g)^{(n)} = f^{(n)} * g$ and

$$x^m = (x-y+y)^m = \sum_{j=0}^{m} \binom{m}{j} (x-y)^j y^{m-j}.$$

~~is a complete metric space. Such a space is called a Fréchet space. Unfortunately, the metric on $\mathcal{S}(\mathbb{R})$ is not induced from any norm, so $\mathcal{S}(\mathbb{R})$ is not a Banach space.~~

~~Expanded discussion of these issues is given in the Appendix.~~

Lemma

$C_c^\infty(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$, in the topology of $\mathcal{S}(\mathbb{R})$. That is, if $f \in \mathcal{S}(\mathbb{R})$ then $\exists f_k \in C_c^\infty(\mathbb{R})$ s.t. $f_k \to f$ in $\mathcal{S}(\mathbb{R})$.

Proof:

Let $\theta \in C_c^\infty(\mathbb{R})$ be any fixed function s.t. $\operatorname{supp}(\theta) \subseteq [-1,1]$ & $\theta(0) = 1$. Define

$$\theta_k(x) = \theta\left(\frac{x}{k}\right).$$

Then $f\theta_k \in C_c^\infty(\mathbb{R})$ & we claim that $f\theta_k \to f$ in $\mathcal{S}(\mathbb{R})$.

To prove this, we must show that

$$\forall m, n \geq 0, \quad \| x^m f^{(n)}(x) - x^m (f\theta_k)^{(n)}(x) \|_\infty \longrightarrow 0$$
as $k \to \infty$.

<u>Case 1: $n=0$</u>

Fix any $m \geq 0$ & any $\varepsilon > 0$. Then $\exists\, k_0 \geq 0$ s.t.

$$\sup_{|x| \geq k_0} |x^m f(x)| < \varepsilon.$$

Hence for $k > k_0$, we have that

$$\|x^m f(x) - x^m (f\theta_k)(x)\|_\infty \leq \sup_{|x| \geq k} |x^m f(x)| < \varepsilon.$$

Thus $\|x^m f(x) - x^m (f\theta_k)(x)\|_\infty \longrightarrow 0$ as $k \to \infty$.

<u>Case 2: $n > 0$</u>

Fix any $n > 0$ & $m \geq 0$. Then by the product rule for derivatives, we have that

$$\| x^m f^{(n)}(x) - x^m (f\theta_k)^{(n)}(x) \|_\infty$$

$$= \| x^m f^{(n)}(x) - x^m \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(x)\, \theta_k^{(n-j)}(x) \|_\infty$$

$$\le \| x^m f^{(n)}(x) - x^m f^{(n)}(x)\, \theta_k(x) \|_\infty$$

$$+ \sum_{j=0}^{n-1} \binom{n}{j} \| x^m f^{(j)}(x)\, \theta_k^{(n-j)}(x) \|_\infty$$

$$= I_1(k) + I_2(k).$$

Now, since $f^{(n)} \in \mathcal{S}(\mathbb{R})$, we have by Case 1 that $I_1(k) \to 0$ as $k \to \infty$. Also,

$$\theta_k^{(j)}(x) = \frac{d^j}{dx^j} \theta\left(\frac{x}{k}\right) = \frac{1}{k^j} \theta'\left(\frac{x}{k}\right),$$

so

$$I_2(k) \le \sum_{j=0}^{n-1} \binom{n}{j} \| x^m f^{(j)}(x) \|_\infty \; \frac{\| \theta^{(n-j)} \|_\infty}{k^{n-j}}$$

$$\longrightarrow 0 \text{ as } k \to \infty. \quad \blacksquare$$

~~Since $f_k' \to f'$ in $\mathcal{S}(\mathbb{R})$, we have that~~

$$\cancel{\langle f_k, g' \rangle = -\langle f_k', g \rangle \to -\langle f', g \rangle = \langle f, g' \rangle.}$$

~~Hence $g' \in \mathcal{S}(\mathbb{R})$.~~

~~Remark (See Folland, Real Analysis, 2nd ed, p. 294)~~

~~Although f' is not bounded by a polynomial, "on average" it is not "too large" because of its rapid oscillations.~~

Motivation

Since $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$, we may expect that

$\mathcal{S}'(\mathbb{R}) = \mathcal{S}(\mathbb{R})^* \subseteq C_c^\infty(\mathbb{R})^* = \mathcal{D}'(\mathbb{R})$, since it should be "easier" for a linear functional to be continuous on a smaller domain than a larger one. However, to make this precise there must be a connection between convergence in $C_c^\infty(\mathbb{R})$ & convergence in $\mathcal{S}(\mathbb{R})$, or, in other words a connection between their respective topologies.

Exercise

Show that if $f_k, f \in C_c^\infty(\mathbb{R})$, then

$$f_k \to f \text{ in } C_c^\infty(\mathbb{R}) \implies f_k \to f \text{ in } \mathcal{S}(\mathbb{R}).$$

Theorem

$\mathcal{S}'(\mathbb{R}) \subsetneq \mathcal{D}'(\mathbb{R})$. More precisely,

$$\mu \longmapsto \mu|_{C_c^\infty(\mathbb{R})}$$

is a continuous injection of $\mathcal{S}'(\mathbb{R})$ into a proper subspace of $\mathcal{D}'(\mathbb{R})$.

Proof:

Suppose $\mu \in \mathcal{S}'(\mathbb{R})$. Then $\mu|_{C_c^\infty(\mathbb{R})} : C_c^\infty(\mathbb{R}) \to \mathbb{C}$ is a linear functional. Further, if $f_k \to f$ in $C_c^\infty(\mathbb{R})$, then by the exercise we have $f_k \to f$ in $\mathcal{S}(\mathbb{R})$, so $\langle f_k, \mu \rangle \to \langle f, \mu \rangle$ since μ is continuous on $\mathcal{S}(\mathbb{R})$. Thus $\mu|_{C_c^\infty(\mathbb{R})} \in \mathcal{D}'(\mathbb{R})$.

Thus $\mathcal{U} : \mu \mapsto \mu|_{C_c^\infty(\mathbb{R})}$ is a well-defined, antilinear map. To show it is injective, we need only show $\ker(\mathcal{U}) = \{0\}$. So, suppose $\mu \in \mathcal{S}'(\mathbb{R})$ & $\mu|_{C_c^\infty(\mathbb{R})} = 0$. Fix any $f \in \mathcal{S}(\mathbb{R})$. By an earlier theorem, $C_c^\infty(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$, so $\exists f_k \in C_c^\infty(\mathbb{R})$ s.t. $f_k \longrightarrow f$ in $\mathcal{S}(\mathbb{R})$.

Then, since μ is continuous & $f_k \in \ker(\mu|_{C_c^\infty(\mathbb{R})})$,

$$0 = \langle f_k, \mu \rangle \longrightarrow \langle f, \mu \rangle.$$

Thus $\langle f, \mu \rangle = 0 \ \forall f \in \mathcal{S}(\mathbb{R})$, so $\mu = 0$.

Next, to show $\mathcal{U}$ is continuous, suppose that $\mu_n \longrightarrow \mu$ in $\mathcal{S}'(\mathbb{R})$. Since convergence in $\mathcal{S}'(\mathbb{R})$ is weak* convergence, we have

$$\forall f \in \mathcal{S}(\mathbb{R}), \quad \langle f, \mu_n \rangle \longrightarrow \langle f, \mu \rangle.$$

But then the same is true for $f \in C_c^\infty(\mathbb{R})$, so

$$\mu_n|_{C_c^\infty(\mathbb{R})} \longrightarrow \mu|_{C_c^\infty(\mathbb{R})}.$$ Thus $\mathcal{U}$ is continuous.

Finally, we already gave an example of a distribution that belongs to $\mathcal{D}'(\mathbb{R})$ but not $\mathcal{S}'(\mathbb{R})$.

Hence $\mathcal{U}$ is not surjective. $\blacksquare$

<u>Remark</u> (See Benedetto, p. 92)

The <u>Riemann zeta function</u> is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 1.$$

It has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole at $s=1$. The <u>Riemann Hypothesis</u>, whose validity is the most celebrated open problem in analytic number theory, is the statement that the complex zeros of $\zeta(s)$ can only occur when $Re(s) = \frac{1}{2}$.

The Weil distribution $W \in \mathcal{D}'(\mathbb{R})$ is defined in [Benedetto, Harmonic Analysis & Applications, p. 92].

Benedetto has shown that:

The Riemann Hypothesis is valid $\iff W \in \mathcal{S}'(\mathbb{R})$.

Exercise

Show that $\mathcal{E}'(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$. Recall that $\mathcal{E}'(\mathbb{R})$

$\mathcal{E}'(\mathbb{R})$, which is the space of compactly supported distributions, was shown in a previous theorem to equal $C^\infty(\mathbb{R})^*$. Show that $\mu \longmapsto \mu|_{\mathcal{S}(\mathbb{R})}$ is a continuous injection of $\mathcal{E}'(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$ that is not surjective.

Exercise

Let $\mu \in \mathcal{S}'(\mathbb{R})$ & $f \in \mathcal{S}(\mathbb{R})$ be given.

a. Show that $f * \mu \in C(\mathbb{R})$.

Hint: Show $T_a f \to f$ in $\mathcal{S}(\mathbb{R})$ as $a \to 0$.

b. Show $f * \mu \in C'(\mathbb{R})$, & $(f * \mu)' = f' * \mu$.

Hint: Show $\dfrac{T_a f - f}{a} \longrightarrow f'$ in $\mathcal{S}(\mathbb{R})$ as $a \to 0$.

c. Show $f * \mu \in C^\infty(\mathbb{R})$.

Remark

$C^\infty(\mathbb{R})$ is not contained in $\mathcal{S}'(\mathbb{R})$, e.g., $e^{x^2} \notin \mathcal{S}'(\mathbb{R})$. However, the next result will show that $f * \mu \in \mathcal{S}'(\mathbb{R})$.

Theorem

If $f \in \mathcal{S}(\mathbb{R})$ & $\mu \in \mathcal{S}'(\mathbb{R})$ then $f * \mu \in C^\infty(\mathbb{R})$ with at most polynomial growth at ∞. Hence $f*\mu \in \mathcal{S}'(\mathbb{R})$.

Proof:

By an earlier result, $\exists M, N \geq 0$ & $C > 0$ s.t.

$$\forall \varphi \in \mathcal{S}(\mathbb{R}), \quad |\langle \varphi, \mu \rangle| \leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \| x^m \varphi^{(n)}(x) \|_\infty .$$

Hence,

$$\begin{aligned} |(f*\mu)(y)| &= |\langle \widetilde{T_y f}, \mu \rangle| \\ &\leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \| x^m (\widetilde{T_y f})^{(n)}(x) \|_\infty \\ &= C \sum_{m=0}^{M} \sum_{n=0}^{N} \| x^m f^{(n)}(y-x) \|_\infty \\ &= C \sum_{m=0}^{M} \sum_{n=0}^{N} \| (y-x)^m f^{(n)}(x) \|_\infty \\ &\leq C \sum_{m=0}^{M} \sum_{n=0}^{N} \sum_{j=0}^{m} \binom{m}{j} |y|^{m-j} \| x^j f^{(n)}(x) \|_\infty \end{aligned}$$

This is a polynomial in $|y|$, so $f*\mu$ has polynomial growth. ■

Exercise

Show that if $f \in \mathcal{S}(\mathbb{R})$ & $\mu \in \mathcal{S}'(\mathbb{R})$, then $f * \mu$ need not be integrable.

Hint: $e^{-x^2} * x^2$.

Exercise

Given $\mu \in \mathcal{S}'(\mathbb{R})$, prove that

$$L : \mathcal{S}(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}), \quad f \longmapsto f * \mu$$

is a linear & continuous map.