Rudin, Chapter 1: Topological Vector Spaces

We've already examined topological vector spaces (TVS) somewhat. Our goal now is to fill in some of the more subtle details, and especially to see how to construct the inductive limit topology on $C_c(\mathbb{R})$ on $C_c^{\infty}(\mathbb{R})$.

We will follow Chapter 1 of Rudin. Many results have already been covered, so we focus on the new definitions & results.

Recall that a <u>TVS</u> is a vector space with a topology that has the property that both vector addition & scalar multiplication are continuous.

Example: A vector space whose topology is induced from a family of seminorms $\{p_\alpha\}_{\alpha \in J}$.

<u>Note</u>
Rudin adds the following requirement to the definition of TVS:

- singletons $\{x\}$ must be closed.

<u>Exercise:</u> Show this implies the topology is Hausdorff.

Recall

For a topology induced from seminorms $\{\rho_\alpha\}_{\alpha \in J}$, we know:

$$\text{Hausdorff} \iff \left(\rho_\alpha(x) = 0 \ \forall \alpha \Rightarrow x = 0\right).$$

Other Rudin notation

For Rudin:

(a) Neighborhood means open neighborhood. That is,

$$U \text{ is a neighborhood of } x \iff U \text{ open}, \ x \in U.$$

I will try to write open neighborhood.

Rudin: Neighborhood means open neighborhood.

Some extra definitions

Bounded Set

(a) A subset E of a TVS X is bounded if

$$\forall \text{ open neighborhood } V \text{ of } 0 \quad \exists s > 0 \text{ s.t.}$$

$$t > s \Rightarrow E \subseteq tV$$

(b) We say that X is locally bounded if 0 has a bounded open neighborhood.

(c) We say X is locally compact if 0 has an open neighborhood whose closure is compact.

Local Base

Motivation: In a TVS we can always translate to be near the origin.

A local base (at 0) for a TVS X is a collection $\mathcal{B}$ of open neighborhoods of 0 such that:

$$U \text{ open}, 0 \in U \Rightarrow \exists B \in \mathcal{B} \text{ s.t. } 0 \in B \subseteq U.$$

Some types of TVS (Rudin's definitions)

(a) A TVS X is an <u>F-space</u> if its topology is induced from a complete translation-invariant metric d.

(b) A TVS X is a <u>Fréchet space</u> if it is a locally convex F-space.

(c) X has the <u>Heine-Borel property</u> if every closed & bounded subset of X is compact.

Some facts (to be proved)

(a) X locally bounded $\Rightarrow$ X has a countable local base

(b) X metrizable $\Leftrightarrow$ X has a countable local base

(c) X normable $\Leftrightarrow$ X is locally convex & locally bounded

(d) X finite-dimensional $\Leftrightarrow$ X is locally compact

(e) locally bounded + Heine-Borel $\Rightarrow$ finite-dimensional

Balanced Sets

Definition.
A set B in a vector space X is *balanced* if

$$\forall\, |c| \le 1, \quad cB \subseteq B.$$

Theorem (see Rudin)
Let X be a vector space.

(a) If $A \subseteq X$ then $\overline{A} = \bigcap \{A + V : V \text{ open}, 0 \in V\}$

(b) $\overline{A} + \overline{B} \subseteq \overline{A+B}$

(c) Y subspace of $X \Rightarrow \overline{Y}$ is a subspace

(d) $C \subseteq X$ is convex $\Rightarrow C^\circ, \overline{C}$ are convex.

(e) $B \subseteq X$ balanced $\Rightarrow \overline{B}$ balanced

(f) $B \subseteq X$ balanced, $0 \in B \Rightarrow B^\circ$ balanced

(g) $E \subseteq X$ bounded $\Rightarrow \overline{E}$ bounded.

Notation: A° = interior of A

$\overline{A}$ = closure of A

Theorem (Rudin)
Let X be a TVS.

(a) Every [redacted] open neighborhood of 0 contains a balanced open neighborhood of 0

(b) Every convex open neighborhood of 0 contains a balanced convex open neighborhood of 0.

Proof
(a) Suppose $\mathcal{U}$ open, $0 \in \mathcal{U}$. Scalar multiplication

is continuous, i.e.

$$\begin{aligned} \cdot : X \times \mathbb{F} &\to X \\ (x,c) &\mapsto cx \end{aligned} \quad \text{is continuous.}$$

Hence $\cdot^{-1}(\mathcal{U})$ must be an open set in $X \times \mathbb{F}$.

Now,
$$\cdot^{-1}(\mathcal{U}) = \{(x,c) : cx \in \mathcal{U}\}$$

and $(0,0) \in \cdot^{-1}(\mathcal{U})$, so there must exist a base

element $\mathcal{B}$ for the topology on $X \times \mathbb{F}$ such that

$(0,0) \in \mathcal{B} \subseteq \cdot^{-1}(\mathcal{U})$. By HW 1, such a base

element has the form $\mathcal{B} = V \times W$ where

$V \subseteq X$ is open & $W \subseteq \mathbb{F}$ is open. WLOG we can take $W = B_\delta(0)$, which is either an open ball in the complex plane if $\mathbb{F} = \mathbb{C}$, or the interval $(-\delta, \delta)$ if $\mathbb{F} = \mathbb{R}$. In any case,

$$x \in V,\ |c| < \delta \Rightarrow (c, x) \in B \Rightarrow cx \in U$$

That is,

$$|c| < \delta \Rightarrow cV \subseteq U.$$

Let

$$H = \bigcup_{|c|<\delta} cV.$$

Then H is an open neighborhood of 0, H is balanced, & $H \subseteq U$.

(b) See Rudin, p. 12. ∎

Corollary

(a) Every TVS has a balanced local base.

(b) Every locally convex TVS has a balanced convex local base.

Proof

(a) $\mathcal{B} = \{U : U \text{ open}, 0 \in U\}$ is a local base. Let

$$\mathcal{B}_{bal} = \{V : V \text{ open}, 0 \in V, V \text{ is balanced}\}.$$

Exercise: Use the preceding theorem to show that $\mathcal{B}_{bal}$ is a local base. ∎

Theorem

Let X be a TVS, and let $V \subseteq X$ be open with $0 \in V$.

(a) If $0 < r_1 < r_2 < \cdots \to \infty$ then

$$X = \bigcup_{n=1}^{\infty} r_n V.$$

(b) Every compact subset of X is bounded.

(c) If $\delta_1 > \delta_2 > \cdots \to 0$ and V is bounded, then $\{\delta_n V\}_{n \in \mathbb{N}}$ is a local base.

Proof:

(a) Fix $x \in X$. The mapping

$$\begin{aligned} \cdot : \mathbb{F} &\to X \\ c &\mapsto cx \end{aligned}$$

is continuous (exercise). Since V is open & $0 \in V$,

$$U = \cdot^{-1}(V) = \{c \in \mathbb{F} : cx \in V\}$$

is an open neighborhood of $0 \in \mathbb{F}$. Hence

$\frac{1}{r_n} \in U$ for all n large enough, which means

$\frac{1}{r_n} x \in V$ for n large. Therefore $x \in r_n V$ for n large.

(b) Let $K \subseteq X$ be convex. To show that K is bounded, we must show that if V is any open neighborhood of 0, then $\exists s > 0$ s.t. $K \subseteq tV$ for all $t > s$.

Given such a V, by a previous theorem $\exists$ balanced open W with $0 \in W \subseteq V$. By part (a), $K \subseteq X = \bigcup_{n=1}^{\infty} nW$. As K is compact, $\exists\, n_1 < \cdots < n_k$ such that

$$K \subseteq n_1 W \cup \cdots \cup n_k W.$$

Recall that balanced means that $cW \subseteq W$ $\forall |c| < 1$. Hence $n_1 W \subseteq n_2 W \subseteq \cdots \subseteq n_k W$, so $K \subseteq n_k W.$ If $t > n_k$, then

$$K \subseteq n_k W \subseteq tW \subseteq tV.$$

Hence K is bounded.

(c) Let U be an open neighborhood of 0.

Since V is bounded, $\exists s > 0$ such that $V \subseteq tU$ for all $t > s$. Let n be large enough that $\frac{1}{\delta_n} > s$. Then $V \subseteq \frac{1}{\delta_n} U$, so $\delta_n V \subseteq U$. Hence $\{\delta_n V\}_{n \in \mathbb{N}}$ is a local base. $\blacksquare$

Exercise

(a) Show that the balanced subsets of $\mathbb{C}$ (w.r.t. scalar field $\mathbb{C}$) are

$$\emptyset,\ \{0\},\ \text{open balls } B_r(0),\ \text{closed balls } \overline{B_r(0)},\ \mathbb{C}$$

In particular, the only unbounded balanced subset of $\mathbb{C}$ is $\mathbb{C}$.

(b) Given a similar characterization of $\mathbb{R}$ w.r.t. the scalar field $\mathbb{R}$.

(c) Show that if we take $\mathbb{F}=\mathbb{R}$, then there are many other balanced subsets of $\mathbb{C}$ or $\mathbb{R}^2$;

(d) Show that balanced subsets of $\mathbb{F}^n$ are connected.

Exercise

Show that a [redacted] subset of a normed space is bounded in the TVS sense if & only if it is bounded in the usual sense.