C

Functional Analysis and Operator Theory

C.1 Linear Operators on Normed Spaces

In this section we will review the basic properties of linear operators on normed spaces.

Definition C.1 (Notation for Operators). Let X, Y be vector spaces, and let $T\colon X \to Y$ be a function mapping X into Y. We write either $T(f)$ or Tf to denote the image under T of an element $f \in X$.

(a) T is *linear* if $T(\alpha f+\beta g) = \alpha T(f)+\beta T(g)$ for every $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$.

(b) T is *antilinear* if $T(\alpha f+\beta g) = \bar{\alpha} T(f)+\bar{\beta} T(g)$ for $f, g \in X$ and $\alpha, \beta \in \mathbb{C}$.

(c) T is *injective* if $T(f) = T(g)$ implies $f = g$.

(d) The *kernel* or *nullspace* of T is $\ker(T) = \{f \in X : T(f) = 0\}$.

(e) The *range* of T is $\mathrm{range}(T) = \{T(f) : f \in X\}$.

(f) The *rank* of T is the vector space dimension of its range, i.e., $\mathrm{rank}(T) = \dim(\mathrm{range}(T))$. In particular, T is *finite-rank* if $\mathrm{range}(T)$ is finite-dimensional.

(g) T is *surjective* if $\mathrm{range}(T) = Y$.

(h) T is a *bijection* if it is both injective and surjective.

We use either the symbol I or I_X to denote the identity map of a space X onto itself.

A mapping between vector spaces is often referred to as an *operator* or a *transformation*, especially if it is linear. We introduce the following terminology for operators on normed spaces.

Definition C.2 (Operators on Normed Spaces). Let X, Y be normed linear spaces, and let $L\colon X \to Y$ be a linear operator.

(a) L is *bounded* if there exists a finite $K \geq 0$ such that

$$\forall\, f \in X, \quad \|Lf\| \leq K\,\|f\|.$$

By context, $\|Lf\|$ denotes the norm of Lf in Y, while $\|f\|$ denotes the norm of f in X.

(b) The *operator norm* of L is

$$\|L\| \;=\; \sup_{\|f\|=1} \|Lf\|. \tag{C.1}$$

On those occasions where we need to specify the spaces in question, we will write $\|L\|_{X\to Y}$ for the operator norm of $L\colon X \to Y$.

(c) We set

$$\mathcal{B}(X,Y) \;=\; \bigl\{L\colon X \to Y \;:\; L \text{ is bounded and linear}\bigr\}.$$

If $X = Y$ then we write $\mathcal{B}(X) = \mathcal{B}(X,X)$.

(f) If $Y = \mathbb{C}$ then we say that L is a *functional*. The set of all bounded linear functionals on X is the *dual space* of X, and is denoted

$$X^* \;=\; \mathcal{B}(X,\mathbb{C}) \;=\; \bigl\{L\colon X \to \mathbb{C} \;:\; L \text{ is bounded and linear}\bigr\}.$$

Another common notation for the dual space is X'.

Notation C.3 (Terminology for Unbounded Operators). Unbounded operators are often not defined on the entire space X but only on some dense subspace. For example, the differentiation operator $Df = f'$ is not defined on all of $L^p(\mathbb{R})$, but it is common to refer to the "differentiation operator D on $L^p(\mathbb{R})$", with the understanding that D is only defined on some associated dense subspace such as $L^p(\mathbb{R}) \cap C^1(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$. Another common terminology is to write that "$D\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is densely defined," again meaning that the domain of D is a dense subspace of $L^p(\mathbb{R})$ and D maps this domain into $L^p(\mathbb{R})$.

Exercise C.4. Let X, Y be normed linear spaces. Let $L\colon X \to Y$ be a linear operator.

(a) L is injective if and only if $\ker L = \{0\}$.

(b) If L is a bijection then the inverse map $L^{-1}\colon Y \to X$ is also a linear bijection.

(c) L is bounded if and only if $\|L\| < \infty$.

(d) If L is bounded then $\|Lf\| \leq \|L\|\,\|f\|$ for every $f \in X$, and $\|L\|$ is the smallest K such that $\|Lf\| \leq K\|f\|$ for all $f \in X$.

(e) $\|L\| = \sup\limits_{\|f\|\leq 1} \|Lf\| = \sup\limits_{f\neq 0} \dfrac{\|Lf\|}{\|f\|}$.

Example C.5. Consider a linear operator on a finite-dimensional space, say $L\colon \mathbb{C}^n \to \mathbb{C}^m$. For simplicity, impose the Euclidean norm on both $\mathbb{C}^n$ and $\mathbb{C}^m$. If we let $C = \{x \in \mathbb{C}^n : \|x\| = 1\}$ be the unit sphere in $\mathbb{C}^n$, then $L(C) = \{Lx : \|x\| = 1\}$ is a (possibly degenerate) ellipsoid in $\mathbb{R}^m$. The supremum in the definition of the operator norm of L is achieved in this case, and is the length of a semimajor axis of the ellipsoid $L(C)$. Thus, $\|L\|$ is the "maximum distortion" of the unit sphere under L, illustrated for the case $m = n = 2$ (with real scalars) in Figure C.1.

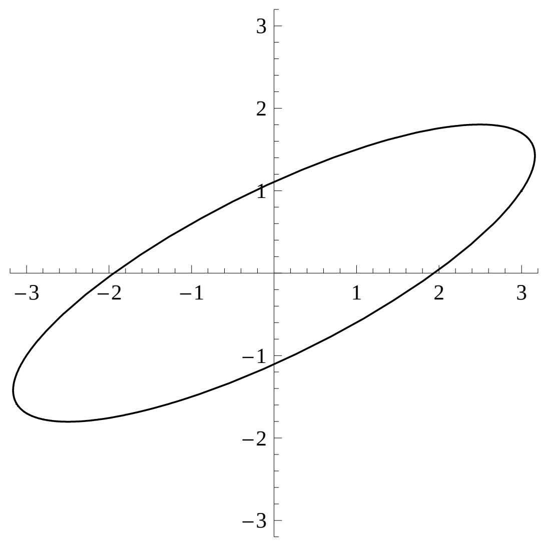

Fig. C.1. Image of the unit circle under a particular linear operator $L\colon \mathbb{R}^2 \to \mathbb{R}^2$. The operator norm $\|L\|$ of L is the length of a semimajor axis of the ellipse.

C.1.1 Equivalence of Boundedness and Continuity

Our first main result of this section shows that continuity is equivalent to boundedness for *linear* operators on normed spaces. Recall that, by Lemma A.53, if X and Y are normed spaces, then $L\colon X \to Y$ is *continuous at a point* $f \in X$ if $f_n \to f$ in X implies $Lf_n \to Lf$ in Y, and L is *continuous* if it is continuous at every point.

Theorem C.6 (Equivalence of Bounded and Continuous Linear Operators). *If X, Y are normed spaces and $L\colon X \to Y$ is linear, then the following statements are equivalent.*

(a) *L is continuous at some $f \in X$.*

(b) *L is continuous at $f = 0$.*

(c) *L is continuous.*

(d) *L is bounded.*

Proof. (c) ⇒ (d). Suppose that L is continuous but unbounded. Then we have $\|L\| = \infty$, so there must exist $f_n \in X$ with $\|f_n\| = 1$ such that $\|Lf_n\| \geq n$. Set $g_n = f_n/n$. Then $\|g_n - 0\| = \|g_n\| = \|f_n\|/n \to 0$, so $g_n \to 0$. Since L is continuous and linear, this implies $Lg_n \to L0 = 0$. By the continuity of the norm, we therefore have $\|Lg_n\| \to \|0\| = 0$. However,

$$\|Lg_n\| \ = \ \frac{1}{n}\,\|Lf_n\| \ \geq \ \frac{1}{n} \cdot n \ = \ 1$$

for all n, which is a contradiction. Hence L must be bounded. □

Thus, if X, Y are normed and $L\colon X \to Y$ is linear, the terms "continuous" and "bounded" are interchangeable.

C.1.2 Isomorphisms

The notion of a topological isomorphism (or homeomorphism) between arbitrary topological spaces was introduced in Definition A.50. We repeat it here for the case of normed spaces, along with additional terminology for operators that preserve norms.

Definition C.7 (Isometries and Isomorphisms). Let X, Y be normed spaces, and let $L\colon X \to Y$ be linear.

(a) If $L\colon X \to Y$ is a linear bijection such that both L and L^{-1} are continuous, then L is called a *topological isomorphism*, or is said to be *continuously invertible*.

(b) If there exists a topological isomorphism $L\colon X \to Y$, then we say that X and Y are *topologically isomorphic*.

(c) If $\|Lf\| = \|f\|$ for all $f \in X$ then L is called an *isometry* or is said to be *norm-preserving*.

(d) An isometry $L\colon X \to Y$ that is a bijection is an *isometric isomorphism*.

(e) If there exists an isometry $L\colon X \to Y$ then we say that X and Y are *isometrically isomorphic*, and we write $X \cong Y$ in this case.

On occasion, we will deal with *antilinear isometric isomorphisms*, which are entirely analogous except that the mapping L is antilinear instead of linear.

Remark C.8. The Inverse Mapping Theorem, which will be discussed in Section C.13, states that if X and Y are Banach spaces and $L\colon X \to Y$ is a bounded linear bijection, then L^{-1} is automatically bounded and hence L is a topological isomorphism. Thus, when X and Y are Banach spaces, every continuous linear bijection is actually a topological isomorphism.

We have the following special terminology for isometric isomorphisms on Hilbert spaces.

Definition C.9 (Unitary Operator). If H, K are Hilbert spaces and $L\colon H \to K$ is an isometric isomorphism, then L is called a *unitary operator*, and in this case we say that H and K are *unitarily isomorphic*.

An isometry on an inner product space must preserve the inner product as well as the norm.

Exercise C.10. Let H, K be inner product spaces, and let $L\colon H \to K$ be a linear mapping. Prove that L is an isometry if and only if $\langle Lf, Lg\rangle = \langle f, g\rangle$ for all f, $g \in H$.

C.1.3 Eigenvalues and Eigenvectors

We recall the definition of the eigenvalues and eigenvectors of an operator that maps a space into itself.

Definition C.11 (Eigenvalues and Eigenvectors). Let X be a normed space and $L\colon X \to X$ a linear operator.

(a) A scalar λ is an *eigenvalue* of L if there exists a nonzero vector $f \in X$ such that $Lf = \lambda f$.

(b) A nonzero vector $f \in X$ is an *eigenvector* of L if there exists a scalar λ such that $Lf = \lambda f$.

If x is an eigenvector of L corresponding to the eigenvalue λ, then we often say that x is a λ-*eigenvector*.

If λ is an eigenvalue of L, then $\ker(L - \lambda I)$ is called the *eigenspace* corresponding to λ, or the λ-*eigenspace* for short.

Additional Problems

C.1. Show that if X is any finite-dimensional vector space (under any norm) and Y is any normed linear space, then every linear function $L\colon X \to Y$ is bounded.

C.2. (a) Define $L\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $L(x) = (x_2, x_3, \dots)$. Prove that this *left-shift operator* is bounded, linear, surjective, not injective, and is not an isometry. Find $\|L\|$ and all eigenvalues and eigenvectors of L.

(b) Define $R\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ by $R(x) = (0, x_1, x_2, x_3, \dots)$. Prove that this *right-shift operator* is bounded, linear, injective, not surjective, and is an isometry. Find $\|R\|$ and all eigenvalues and eigenvectors of R.

(c) Compute LR and RL and show that $LR \neq RL$. Contrast this computation with the fact that in finite dimensions, if A, $B\colon \mathbb{C}^n \to \mathbb{C}^n$ are linear maps (hence correspond to multiplication by $n \times n$ matrices), then $AB = I$ if and only if $BA = I$.

C.3. If X, Y are normed spaces and $L\colon X \to Y$ is continuous, show that $\ker(L)$ is a closed subspace of X.

C.4. Let X be a Banach space and Y a normed linear space. Suppose that $L\colon X \to Y$ is bounded and linear. Prove that if there exists $c > 0$ such that $\|Lf\| \geq c\|f\|$ for all $f \in X$, then L is injective and $\mathrm{range}(L)$ is closed.

C.5. Show that if $L\colon X \to Y$ is a topological isomorphism, then we have $\|L^{-1}\|^{-1}\,\|f\| \leq \|Lf\| \leq \|L\|\,\|f\|$ for all $f \in X$.

C.6. Show that if H, K are separable Hilbert spaces, then H and K are unitarily isomorphic.

C.7. Let A be an $m \times n$ complex matrix, which we view as a linear transformation $A\colon \mathbb{C}^n \to \mathbb{C}^m$. The operator norm of A depends on the choice of norm for $\mathbb{C}^n$ and $\mathbb{C}^m$. Compute an explicit formula for $\|A\|$, in terms of the entries of A, when the norm on $\mathbb{C}^n$ and $\mathbb{C}^m$ is taken to be the ℓ^1 norm. Then do the same for the ℓ^∞ norm. Compare your formulas to the version of Schur's Test given in Theorem C.20.

C.8. The Axiom of Choice implies that every vector space X has a Hamel basis (Theorem G.3). Use this to show that if X is an infinite-dimensional normed linear space, then there exists a linear functional $\mu\colon X \to \mathbb{C}$ that is unbounded.

C.2 Some Useful Operators

In this section we describe several types of operators that appear often in the main part of the text.

C.2.1 Orthogonal Projections

We begin with orthogonal projections in Hilbert spaces.

Definition C.12 (Orthogonal Projection). Let M be a closed subspace of a Hilbert space H. Define $P\colon H \to H$ by $Ph = p$, where p is the orthogonal projection of h onto M, (see Definition A.97). The operator P is the *orthogonal projection of H onto M*.

Exercise C.13 (Properties of Orthogonal Projections). Let $M \neq \{0\}$ be a closed subspace of a Hilbert space H, and let P be the orthogonal projection of H onto M. Show that the following statements hold.

(a) If $h \in H$ then Ph is the unique vector in M such that $h - Ph \in M^\perp$.

(b) $\|h - Ph\| = \mathrm{dist}(h, M)$ for every $h \in H$.

(c) P is linear, $\|Ph\| \leq \|h\|$ for every $h \in H$, and $\|P\| = 1$.

(d) P is *idempotent*, i.e., $P^2 = P$.

(e) $\ker(P) = M^\perp$ and $\mathrm{range}(P) = M$.

(f) $I - P$ is the orthogonal projection of H onto $M^\perp$.

C.2.2 Multiplication Operators

Next we consider two types of "multiplication" operators. The first type multiplies each term in an orthonormal basis expansion by a fixed scalar.

Exercise C.14. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a separable Hilbert space H. Then, by Exercise A.103, we know that every $f \in H$ can be written $f = \sum_{n=1}^{\infty} \langle f, e_n\rangle\, e_n$. Fix any sequence of scalars $\lambda = (\lambda_n)_{n\in\mathbb{N}}$. For those $f \in H$ for which the following series converges, define

$$M_\lambda f \;=\; \sum_{n=1}^{\infty} \lambda_n \langle f, e_n\rangle\, e_n. \tag{C.2}$$

Prove the following facts.

(a) The series defining $M_\lambda f$ in (C.2) converges for every $f \in H$ if and only if $\lambda \in \ell^\infty$. In this case M_λ is a bounded linear mapping of H into itself, and $\|M_\lambda\| = \|\lambda\|_\infty$.

(b) If $\lambda \notin \ell^\infty$, then M_λ defines an unbounded linear mapping from

$$\operatorname{domain}(M_\lambda) \;=\; \Big\{ f \in H : \sum_{n=1}^{\infty} |\lambda_n \langle f, e_n\rangle|^2 < \infty \Big\} \tag{C.3}$$

into H. Note that $\operatorname{domain}(M_\lambda)$ contains the finite span of $\{e_n\}_{n\in\mathbb{N}}$, and hence is dense in H.

If $H = \ell^2$ and $\{e_n\}_{n\in\mathbb{N}}$ is the standard basis for ℓ^2, then the multiplication operator M_λ defined in equation (C.2) is simply componentwise multiplication: $M_\lambda x = \lambda x = (\lambda_1 x_1, \lambda_2 x_2, \dots)$ for $x = (x_1, x_2, \dots) \in \ell^2$. This is a discrete version of the multiplication operator defined in the next exercise.

Exercise C.15. Let $\phi\colon \mathbb{R} \to \mathbb{C}$ and $1 \le p \le \infty$ be given.

(a) Show that if $\phi \in L^\infty(\mathbb{R})$, then $M_\phi f = f\phi$ is a bounded mapping of $L^p(\mathbb{R})$ into itself, and $\|M_\phi\| = \|\phi\|_\infty$.

(b) Conversely, show that if $f\phi \in L^p(\mathbb{R})$ for every $f \in L^p(\mathbb{R})$, then we must have $\phi \in L^\infty(\mathbb{R})$.

C.2.3 Integral Operators

Now we define the important class of *integral operators* for the setting of the real line.

Definition C.16 (Integral Operator). Let k be a fixed measurable function on $\mathbb{R}^2$. Then the *integral operator L_k with kernel k* is formally defined by

$$L_k f(x) \;=\; \int k(x,y)\, f(y)\, dy, \tag{C.4}$$

i.e., $L_k f$ is defined whenever this integral makes sense.

An integral operator is a generalization of ordinary matrix-vector multiplication. Let A be an $m \times n$ matrix with entries a_{ij} and let $u \in \mathbb{C}^n$ be given. Then $Au \in \mathbb{C}^m$, and its components are

$$(Au)_i \;=\; \sum_{j=1}^{n} a_{ij}\, u_j, \qquad i = 1, \dots, m.$$

Thus, the function values $k(x,y)$ are analogous to the entries a_{ij} of the matrix A, and the values $L_k f(x)$ are analogous to the entries $(Au)_i$.

Example C.17 (Tensor Product Kernels). The *tensor product* of two functions g, h on $\mathbb{R}$ is the function $g \otimes h$ on $\mathbb{R}^2$ defined by

$$(g \otimes h)(x,y) \;=\; g(x)\,\overline{h(y)}, \qquad x, y \in \mathbb{R}.$$

Sometimes the complex conjugate is omitted in the definition of tensor product, but it will be convenient for our purposes to include it.

An important special case of an integral operator is where the kernel k is a tensor product. If we assume that g, $h \in L^2(\mathbb{R})$ and set $k = g \otimes h$, then for $f \in L^2(\mathbb{R})$,

$$L_k f(x) \;=\; \int g(x)\,\overline{h(y)}\, f(y)\, dy \;=\; \langle f, h\rangle\, g(x),$$

at least for all x for which $g(x)$ is defined. If either $g = 0$ or $h = 0$ then L_k is the zero operator, otherwise the range of L_k is the one-dimensional subspace spanned by g. Thus, L_k is a very "simple" operator in this case, being a bounded, rank one operator on $L^2(\mathbb{R})$.

Notation C.18. When $k = g \otimes h$ is a tensor product, we often identify the operator L_k with the kernel $g \otimes h$. In other words, given g and h we often let $g \otimes h$ denote the operator whose rule is

$$(g \otimes h)(f) \;=\; \langle f, h\rangle\, g, \qquad f \in L^2(\mathbb{R}).$$

We can extend this notion of an operator $g \otimes h$ to arbitrary Hilbert spaces by simply replacing $L^2(\mathbb{R})$ with H on the line above. That is, if g, $h \in H$ then we define $g \otimes h$ to be the rank one operator given by $(g \otimes h)(f) = \langle f, h\rangle\, g$ for $f \in H$. Note that if $g = h$ and $\|g\|_2 = 1$, then $g \otimes g$ is the orthogonal projection of H onto the line through g.

In general, it is not obvious how to tie properties of the kernel k to properties of the corresponding integral operator L_k. The next two theorems will provide sufficient conditions that imply L_k is a bounded operator on $L^2(\mathbb{R})$. First, we show that if the kernel is square-integrable, then the corresponding integral operator is a bounded mapping of $L^2(\mathbb{R})$ into itself. The *Hilbert–Schmidt operators* on $L^2(\mathbb{R})$ are precisely those operators that can be written as integral operators with kernels $k \in L^2(\mathbb{R}^2)$, see Theorem C.79.

Theorem C.19 (Hilbert–Schmidt Integral Operators). *Let $k \in L^2(\mathbb{R}^2)$ be fixed. Then the integral operator L_k given by* (C.4) *defines a bounded mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|L_k\| \le \|k\|_2$.*

Proof. Suppose that $k \in L^2(\mathbb{R}^2)$, and define $k_x(y) = k(x,y)$. Then, by Fubini's Theorem, $k_x \in L^2(\mathbb{R})$ for a.e. x. Hence, if $f \in L^2(\mathbb{R})$, then

$$L_k f(x) \;=\; \langle k_x, \bar{f}\,\rangle \;=\; \int k_x(y)\, f(y)\, dy$$

exists for almost every x.

To see why $L_k f$ is a measurable and square-integrable function of x, consider first the case where f and k are both nonnegative. Then $k(x,y)\, f(y)$ is a measurable function on $\mathbb{R}^2$, so Tonelli's Theorem tells us that $L_k f(x) = \int k(x,y)\, f(y)\, dy$ is a measurable function of x. We estimate its L^2-norm by applying the Cauchy–Bunyakowski–Schwarz Inequality:

$$\begin{aligned}
\|L_k f\|_2^2 \;&=\; \int |L_k f(x)|^2\, dx \\
&=\; \int \left| \int k(x,y)\, f(y)\, dy \right|^2 dx \\
&\le\; \int \left(\int |k(x,y)|^2\, dy \right) \left(\int |f(y)|^2\, dy \right) dx \\
&=\; \int \int |k(x,y)|^2\, dy\, \|f\|_2^2\, dx \\
&=\; \|k\|_2^2\, \|f\|_2^2 \;<\; \infty.
\end{aligned}$$

Hence $L_k f \in L^2(\mathbb{R})$.

Now suppose that $f \in L^2(\mathbb{R})$ and $k \in L^2(\mathbb{R}^2)$ are arbitrary, and write $f = (f_1^+ - f_1^-) + i(f_2^+ - f_2^-)$ and $k = (k_1^+ - k_1^-) + i(k_2^+ - k_2^-)$, where each function $f_\ell^\pm$ and $k_j^\pm$ is nonnegative. Then, by the work above, each function $L_{k_j^\pm}(f_\ell^\pm)$ is measurable and belongs to $L^2(\mathbb{R})$. Since $L_k f$ is a finite linear combination of the sixteen functions $L_{k_j^\pm}(f_\ell^\pm)$, we conclude that $L_k f$ is measurable and belongs to $L^2(\mathbb{R})$.

Now that we know that $L_k f$ is measurable, we can follow exactly the same estimates as were used in the nonnegative case to show that $\|L_k f\|_2 \le \|k\|_2\, \|f\|_2$. Hence L_k is a bounded mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|L_k\| \le \|k\|_2$. □

The next result, originally formulated in [Sch11], is often called *Schur's Test* (not to be confused with *Schur's Lemma*). Here we formulate Schur's Test for boundedness of integral operators, but it is instructive to compare this result to Problem C.7, which essentially is Schur's Test for finite matrices.

Theorem C.20 (Schur's Test). *Assume that k is a measurable function on $\mathbb{R}^2$ that satisfies the mixed-norm conditions*

$$\begin{aligned} C_1 &= \operatorname*{ess\,sup}_{x\in\mathbb{R}} \int |k(x,y)|\,dy \;<\; \infty, \\ C_2 &= \operatorname*{ess\,sup}_{y\in\mathbb{R}} \int |k(x,y)|\,dx \;<\; \infty. \end{aligned} \tag{C.5}$$

Then the integral operator L_k given by (C.4) *defines a bounded mapping of $L^2(\mathbb{R})$ into itself, with operator norm $\|L_k\| \le (C_1C_2)^{1/2}$.*

Proof. As in the proof of Theorem C.19, measurability of L_kf is most easily shown by first considering nonnegative f, k, and then extending to the general case. We omit the details and assume that L_kf is measurable for all $f \in L^2(\mathbb{R})$. Then, by applying the Cauchy–Bunyakowski–Schwarz Inequality, we have

$$\begin{aligned} \|L_kf\|_2^2 &= \int |L_kf(x)|^2\,dx \\ &= \int \left| \int k(x,y)\,f(y)\,dy \right|^2 dx \\ &\le \int \left(\int |k(x,y)|^{1/2} \cdot |k(x,y)|^{1/2}\,|f(y)|\,dy \right)^2 dx \\ &\le \int \left(\int |k(x,y)|\,dy \right) \left(\int |k(x,y)|\,|f(y)|^2\,dy \right) dx \\ &\le \int C_1 \int |k(x,y)|\,|f(y)|^2\,dy\,dx \\ &= C_1 \int |f(y)|^2 \int |k(x,y)|\,dx\,dy \\ &\le C_1 \int |f(y)|^2\,C_2\,dy \;=\; C_1C_2\,\|f\|_2^2, \end{aligned}$$

where we have used Tonelli's Theorem to interchange the order of integration. Thus L_k is bounded and $\|L_k\| \le (C_1C_2)^{1/2}$. □

The next exercise shows that the hypotheses of Schur's Test actually yield boundedness on every $L^p(\mathbb{R})$, not just for $p = 2$.

Exercise C.21. Show that if k satisfies the conditions (C.5), then L_k is a bounded mapping of $L^p(\mathbb{R})$ into itself for every $1 \le p \le \infty$, with $\|L_k\| \le C_1^{1/p'}C_2^{1/p}$.

Remark C.22. If we assume only that k is measurable and that $C_2 < \infty$ (with no hypothesis about C_1), then we have that $L_k\colon L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is a bounded mapping. Similarly, if $C_1 < \infty$ then $L_k\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ is bounded. Further, the proofs of these two particular "endpoint cases" are quite simple. Exercise C.21 says that if C_1 and C_2 are both finite, then not only do we have boundedness for the straightforward endpoint cases $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$, but we can also prove the more difficult result of boundedness on $L^p(\mathbb{R})$ for each $1 \le p \le \infty$. This type of extension problem is very common, and indeed there is an entire theory of *interpolation theorems* that deal with similar extension issues, see [BeL76]. One basic interpolation theorem is the *Riesz–Thorin Theorem*, which is discussed in Section 2.3.

C.2.4 Convolution

Convolution is considered in detail in Section 1.3. Here we give another view of convolution by considering it to be a special type of an integral operator. In particular, the convolution of f and g is $(f * g)(x) = \int f(y)\, g(x-y)\, dy$, whenever this is defined. With g fixed, the mapping $f \mapsto f * g$ is the integral operator L_k whose kernel is $k(x,y) = g(x-y)$.

Exercise C.23. Use Schur's Test to prove the following version of Young's Inequality (compare Exercise 1.25): If $1 \le p \le \infty$, then

$$\forall\, f \in L^p(\mathbb{R}), \quad \forall\, g \in L^1(\mathbb{R}), \quad \|f * g\|_p \le \|f\|_p\, \|g\|_1.$$

As a consequence, $L^1(\mathbb{R})$ is closed under convolution and is an example of a *Banach algebra* (see Definition C.28).

Additional Problems

C.9. Choose $\lambda \in \ell^\infty$, and set $\delta = \inf_n |\lambda_n|$. Define M_λ as in Exercise C.14, and prove the following.

(a) Each λ_n is an eigenvalue for M_λ with corresponding eigenvector e_n.

(b) M_λ is injective if and only if $\lambda_n \ne 0$ for every n.

(c) M_λ is surjective if and only if $\delta > 0$.

(d) If $\delta = 0$ but $\lambda_n \ne 0$ for every n then range(M_λ) is a dense but proper subspace of H.

(e) M_λ is unitary if and only if $|\lambda_n| = 1$ for every n.

C.10. Let $\phi \in L^\infty(\mathbb{R})$ be fixed, and let M_ϕ be defined as in Exercise C.15. Fix $1 \le p \le \infty$.

(a) (a) Determine a necessary and sufficient condition on ϕ that implies that $M_\phi\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is injective.

(b) Determine a necessary and sufficient condition on ϕ that implies that $M_\phi\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is surjective.

(c) Show directly that if M_ϕ is injective but not surjective then the inverse mapping $M_\phi^{-1}\colon \mathrm{range}(M_\phi) \to L^p(\mathbb{R})$ is unbounded.

C.3 The Space $\mathcal{B}(X,Y)$

Now we turn our attention to the space $\mathcal{B}(X,Y)$ of all bounded linear maps from X into Y, which was introduced in Definition C.2.

Exercise C.24. Let X and Y be normed spaces.

(a) $\mathcal{B}(X,Y)$ is a vector space, and the operator norm is a norm on $\mathcal{B}(X,Y)$.

(b) If Y is a Banach space, then $\mathcal{B}(X,Y)$ is a Banach space with respect to operator norm.

Consequently, if X is any normed space, then its dual space $X^* = \mathcal{B}(X,\mathbb{C})$ is a Banach space.

In addition to operations of vector addition and scalar multiplication, there is a third operation that we can perform with operators, namely composition.

Exercise C.25. Prove that the operator norm is *submultiplicative*, i.e., if $A \in \mathcal{B}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$, then $BA \in \mathcal{B}(X,Z)$, and

$$\|BA\| \;\le\; \|B\|\,\|A\|. \tag{C.6}$$

In particular, $\mathcal{B}(X)$ is closed under compositions, and is an example of a noncommutative *Banach algebra* (see Definition C.28).

The following useful exercise shows that a bounded operator that is defined on a dense subspace of a normed space can be extended to the entire space.

Exercise C.26 (Extension of Bounded Operators). Let Y be a dense subspace of a normed space X, and let Z be a Banach space. Let $L \in \mathcal{B}(Y,Z)$ be given.

(a) Show that there exists a unique operator $\widetilde{L} \in \mathcal{B}(X,Z)$ whose restriction to Y is L. Prove that $\|\widetilde{L}\| = \|L\|$.

(b) Show that if $L\colon Y \to \mathrm{range}(L)$ is a topological isomorphism, then $\widetilde{L}\colon X \to \overline{\mathrm{range}(L)}$ is a topological isomorphism.

C.4 Banach Algebras

We have seen some examples of Banach spaces that, in addition to being complete normed vector spaces, are also are closed under an additional "multiplication-like" operation. These are examples of Banach algebras, the precise definition of which is as follows.

Definition C.27 (Algebra). An *algebra* over a field K is a vector space $\mathcal{A}$ over K such that for each x, $y \in \mathcal{A}$ there exists a unique product $xy \in \mathcal{A}$ that satisfies the following for all x, y, $z \in \mathcal{A}$ and $\alpha \in K$:

(a) $(xy)z = x(yz)$,

(b) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$, and

(c) $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

If $K = \mathbb{R}$ then $\mathcal{A}$ is a *real algebra*; if $K = \mathbb{C}$ then $\mathcal{A}$ is a *complex algebra*.

If $xy = yx$ for all $x, y \in \mathcal{A}$ then $\mathcal{A}$ is *commutative*.

If there exists an element $e \in \mathcal{A}$ such that $ex = xe = x$ for every $x \in \mathcal{A}$ then $\mathcal{A}$ is an *algebra with identity*.

Definition C.28 (Banach Algebra). A *normed algebra* is a normed linear space $\mathcal{A}$ that is an algebra and also satisfies

$$\forall\, x, y \in \mathcal{A}, \quad \|xy\| \;\le\; \|x\|\,\|y\|.$$

A *Banach algebra* is a normed algebra that is a Banach space, i.e., it is a complete normed algebra.

Here are the examples of Banach algebras that we have seen so far in this appendix, plus some other examples from Section 1.3.

Exercise C.29. (a) $L^1(\mathbb{R})$ is a commutative Banach algebra under convolution. However, it does not have an identity (see Exercise 1.27).

(b) If X is a Banach space, then $\mathcal{B}(X)$ is a noncommutative Banach algebra with identity under composition of operators.

(c) $C_b(\mathbb{R})$ is a commutative Banach algebra with identity under the operation of pointwise products of functions, i.e., $(fg)(x) = f(x)g(x)$.

(d) $C_0(\mathbb{R})$ is a commutative Banach algebra without identity under the operation of pointwise products of functions.

As in abstract ring theory, the concept of an ideal plays an important role in the theory of Banach algebras. Ideals are the black holes of the algebra, sucking any product of an algebra element with an ideal element into the ideal.

Definition C.30 (Ideals). Let $\mathcal{A}$ be a Banach algebra.

(a) A subspace I of $\mathcal{A}$ is a *left ideal* in $\mathcal{A}$ if $xy \in I$ whenever $x \in \mathcal{A}$ and $y \in I$.

(b) A subspace I of $\mathcal{A}$ is a *right ideal* in $\mathcal{A}$ if $yx \in I$ whenever $x \in \mathcal{A}$ and $y \in I$.

(c) A subspace I of $\mathcal{A}$ is a *two-sided ideal*, or simply an *ideal* in $\mathcal{A}$ if $xy, yx \in I$ whenever $x \in \mathcal{A}$ and $y \in I$.

For example, the space $C_c(\mathbb{R})$ is an ideal in $C_0(\mathbb{R})$ under the operation of pointwise multiplication of functions. By Exercise A.63, we also know that $C_c(\mathbb{R})$ is a dense subspace of $C_0(\mathbb{R})$. However, not all ideals are dense subspaces. For example, if $E \subseteq \mathbb{R}$, then $I = \{f \in C_0(\mathbb{R}) : f(x) = 0 \text{ for all } x \in E\}$ is a proper, closed ideal in $C_0(\mathbb{R})$.

Exercise C.31. Let I be an ideal in a commutative Banach algebra $\mathcal{A}$.

(a) Prove that if $x \in \mathcal{A}$, then $x\mathcal{A} = \{xy : y \in \mathcal{A}\}$ is an ideal in $\mathcal{A}$, called the *ideal generated by* x.

(b) Give a specific example that shows that x need not belong to $x\mathcal{A}$.

(c) Show that if I is an ideal in $\mathcal{A}$, then so is its closure $\overline{I}$.

Some Banach algebras also have an additional operation that has properties similar to that of conjugation of complex numbers.

Definition C.32 (Involution). An *involution* on a Banach algebra $\mathcal{A}$ is a mapping $x \mapsto x^*$ of $\mathcal{A}$ into itself that satisfies the following for all x, $y \in \mathcal{A}$ and all scalars $\alpha \in \mathbb{C}$:

(a) $(x^*)^* = x$,

(b) $(xy)^* = y^*x^*$,

(c) $(x+y)^* = x^* + y^*$, and

(d) $(\alpha x)^* = \bar{\alpha}x^*$.

Exercise C.33. Given $f \in L^1(\mathbb{R})$, define $\widetilde{f}(x) = \overline{f(-x)}$. Show that $f \mapsto \widetilde{f}$ defines an involution on $L^1(\mathbb{R})$ with respect to convolution.

Another example of an involution is the adjoint operation on $\mathcal{B}(H)$, see Section C.6 below.

C.5 Some Dual Spaces

In this section we consider the dual space of a Hilbert space and the dual space of the Lebesgue space $L^p(E)$.

C.5.1 The Dual of a Hilbert Space

If H is a Hilbert space and $g \in H$ is fixed, then the Cauchy–Bunyakowski–Schwarz Inequality implies that $\mu_g \colon H \to \mathbb{C}$ given by $\mu_g \colon f \mapsto \langle f, g\rangle$ is a bounded linear functional on H. The *Riesz Representation Theorem* for Hilbert spaces asserts that every bounded linear functional has this form. Consequently, every Hilbert space is "self-dual."

Exercise C.34 (Riesz Representation Theorem). Given $g \in H$, define $\mu_g \colon H \to \mathbb{C}$ by $\mu_g \colon f \mapsto \langle f, g\rangle$.

(a) Show that $\mu_g \in H^*$ for each $g \in H$, and that

$$\|g\| = \|\mu_g\| = \sup_{\|f\|=1} |\langle f, g\rangle|.$$

(b) For each $\mu \in H^*$, show there exists a unique $g \in H$ such that $\mu = \mu_g$.

(c) Define $T\colon H \to H^*$ by $T(g) = \mu_g$. Prove that T is an antilinear isometric bijection of H onto H^*. In particular, $\mu_{\alpha g+\beta h} = \bar{\alpha}\mu_g + \bar{\beta}\mu_h$.

For the specific case of ℓ^2 or $L^2(E)$, the Riesz Representation Theorem takes the following form.

Corollary C.35. (a) *If μ is a bounded linear functional on $\ell^2(I)$, then there exists a unique $y = (y_k)_{k\in I} \in \ell^2(I)$ such that*

$$\mu\colon x \;\mapsto\; \sum_{k\in I} x_k \bar{y}_k \;=\; \langle x, y\rangle, \qquad x = (x_k)_{k\in I} \in \ell^2(I). \tag{C.7}$$

(b) *If μ is a bounded linear functional on $L^2(E)$, then exists a unique $g \in L^2(E)$ such that*

$$\mu\colon f \;\mapsto\; \int_E f(x)\,\overline{g(x)}\,dx \;=\; \langle f, g\rangle, \qquad f \in L^2(E). \tag{C.8}$$

We usually identify the functional $\mu \in H^*$ with the element $g \in H$ that satisfies $\mu = \mu_g$. However, it is important to note that this identification $g \mapsto \mu_g$ is *antilinear.* On the other hand, the examples given in equations (C.7) and (C.8) illustrate that this antilinearity is a natural consequence of the definition of the inner product. For this reason, it is most convenient for us to consider the pairing of a vector f in a normed space X with a linear functional μ on X to be a generalization of the inner product on a Hilbert space, i.e., it is a sesquilinear form that is linear as a function of f but antilinear as a function of μ. We therefore adopt the following notations for denoting the action of a linear functional on a vector.

Notation C.36 (Notation for Linear Functionals). Let X be a normed linear space. Given a fixed linear functional $\mu\colon X \to \mathbb{C}$, we use two notations to denote the image of f under μ.

(a) We write

$$\mu(f)$$

to denote the image of f under μ, with the understanding that this notation is linear in both f and μ, i.e.,

$$\mu(\alpha f + \beta g) \;=\; \alpha\mu(f) + \beta\mu(g)$$

and

$$(\alpha\mu + \beta\nu)(f) \;=\; \alpha\mu(f) + \beta\nu(f).$$

(b) We write

$$\langle f, \mu\rangle$$

to denote the image of f under μ, with the understanding that this notation is linear in f but antilinear in μ, i.e.,

$$\langle \alpha f + \beta g, \mu \rangle \;=\; \alpha \langle f, \mu \rangle + \beta \langle g, \mu \rangle$$

while

$$\langle f, \alpha\mu + \beta\nu \rangle \;=\; \bar{\alpha} \langle f, \mu \rangle + \bar{\beta} \langle f, \nu \rangle. \tag{C.9}$$

This will be the preferred notation throughout this volume.

C.5.2 The Dual of $L^p(E)$

The fact that the dual space of the Hilbert space $L^2(E)$ is (antilinearly) isomorphic to $L^2(E)$ has a generalization to other L^p spaces. By Hölder's Inequality, if $g \in L^{p'}(E)$ is fixed, then $\langle f, \mu_g \rangle = \int_E f(x)\,\overline{g(x)}\,dx$ defines a bounded linear functional μ_g on $L^p(E)$, and the following exercise shows that the operator norm of μ_g equals the $L^{p'}$-norm of the function g.

Exercise C.37. Let E be a Lebesgue measurable subset of $\mathbb{R}$, and fix $1 \le p \le \infty$. For each $g \in L^{p'}(E)$, define $\mu_g \colon L^p(E) \to \mathbb{C}$ by

$$\langle f, \mu_g \rangle \;=\; \int_E f(x)\,\overline{g(x)}\,dx, \qquad f \in L^p(E). \tag{C.10}$$

Show that $\mu_g \in L^p(E)^*$ and $\|\mu_g\| = \|g\|_{p'}$.

Although we will not prove it, the next theorem states that if $1 \le p < \infty$ then every bounded linear functional on $L^p(E)$ has the form μ_g for some $g \in L^{p'}(E)$. Consequently, $L^p(E)^*$ and $L^{p'}(E)$ are (antilinearly) isomorphic. The standard proof of Theorem C.38 relies on the Radon–Nikodym Theorem (see Theorem D.54).

Theorem C.38 (Dual Space of $L^p(E)$). *Let E be a Lebesgue measurable subset of $\mathbb{R}$, and fix $1 \le p < \infty$. For each $g \in L^{p'}(E)$, define μ_g as in equation* (C.10)*. Then the mapping $T \colon L^{p'}(E) \to L^p(E)^*$ defined by $T(g) = \mu_g$ is an antilinear isometric isomorphism of $L^{p'}(E)$ onto $L^p(E)^*$.*

Remark C.39. Theorem C.38 generalizes to $L^p(X)$ for any measure space (X, μ, Σ) when $1 < p < \infty$. It also generalizes to $L^1(X)$ if μ is σ-finite, see [Fol99] for details. In particular, an analogue of Theorem C.38 holds for the ℓ^p spaces.

If $p = \infty$, then the map $T \colon L^1(E) \to L^\infty(E)^*$ given by $T(g) = \mu_g$ is still an antilinear isometry, but it is not surjective. In this sense, $L^1(E)$ has a canonical image within $L^\infty(E)^*$, but there are functionals in $L^\infty(E)^*$ that do not correspond to elements of $L^1(E)$, compare Problem E.8.

Because of Theorem C.38, we usually identify $L^{p'}(E)$ with $L^p(E)^*$ when p is finite, and also identify $L^1(E)$ with its image in $L^\infty(E)^*$. Abusing notation, we write

$$L^p(E)^* \;=\; L^{p'}(E) \quad \text{for } 1 \le p < \infty,$$

and

$$L^1(E) \;\subseteq\; L^\infty(E)^*,$$

with the understanding that these hold in the sense of the identification of $g \in L^{p'}(E)$ with $\mu_g \in L^p(E)^*$.

C.5.3 The Relation between $L^{p'}(E)$ and $L^p(E)^*$

We have chosen to consider the relation between $L^{p'}(E)$ and $L^p(E)^*$ in a manner that most directly generalizes the inner product on a Hilbert space and the characterization of the dual space of a Hilbert space as given by the Riesz Representation Theorem. With our choice, we write the action of $\mu \in L^p(E)^*$ on $f \in L^p(E)$ as $\langle f, \mu \rangle$, and regard this as a sesquilinear form, linear in f but antilinear in μ. With this notation, the following statements hold (we restrict our attention in this discussion to $1 \le p < \infty$).

(a) $L^p(E)$, $L^{p'}(E)$, and $L^p(E)^*$ are linear spaces.

(b) $L^p(E)^*$ is the space of bounded linear functionals on $L^p(E)$.

(c) $T\colon L^{p'}(E) \to L^p(E)^*$ given by $T(g) = \mu_g$ is an isometric isomorphism, but is antilinear.

To illustrate one advantage of this approach, consider the special case $p = 2$. Since $L^2(E)$ is both a Hilbert space and a particular L^p space, we have introduced two different uses of the notation $\langle \cdot, \cdot \rangle$ with regard to $L^2(E)$. On the one hand, $\langle f, g \rangle$ denotes the inner product of $f, g \in L^2(E)$, while, on the other hand, $\langle f, \mu \rangle$ denotes the action of $\mu \in L^2(E)^*$ on $f \in L^2(E)$. Fortunately, $\langle f, g \rangle = \langle f, \mu_g \rangle$, so our linear functional notation is not in conflict with our inner product notation. This notationally simplifies certain calculations. For example, if $A\colon L^2(E) \to L^2(E)$ is unitary then we have for $f, g \in L^2(E)$ that $\langle f, g \rangle = \langle Af, Ag \rangle$, and also $\langle f, \mu_g \rangle = \langle Af, \mu_{Ag} \rangle$. However, we do have to accept that our identification of g with μ_g is antilinear rather than linear.

There are various alternative approaches, each with their own advantages and disadvantages, that we discuss now.

A second choice is to base our notation on the usual convention that if ν is a linear functional, then the notation $\nu(f)$ is linear in both f and ν. If we follow this convention, then we associate a function $g \in L^{p'}(E)$ with the functional $\nu_g\colon L^p(E) \to \mathbb{C}$ defined by

$$\nu_g(f) \;=\; \int_E f(x)\, g(x)\, dx.$$

Using this notation, the following facts hold.

(a) $L^p(E)$, $L^{p'}(E)$, and $L^p(E)^*$ are linear spaces.

(b) $L^p(E)^*$ is the space of bounded linear functionals on $L^p(E)$.

(c) $U\colon L^{p'}(E) \to L^p(E)^*$ given by $U(g) = \nu_g$ is an isometric isomorphism, and is linear.

This is a natural choice except for the fact that the notation $\nu(f)$ is not an extension of the inner product on $L^2(E)$. Specifically, although we identify $g \in L^2(E)$ with $\nu_g \in L^2(E)^*$, the inner product $\langle f, g\rangle$ does not coincide with $\nu_g(f)$. Hence for $p = 2$, if $A\colon L^2(E) \to L^2(E)$ is unitary, then while we do have $\langle f, g\rangle = \langle Af, Ag\rangle$, we do *not* have equality of $\nu_g(f)$ and $\nu_{Ag}(Af)$. Another consequence is that if $L\colon L^2(E) \to L^2(E)$ is linear, then the adjoint L^* of L defined by the requirement that $\langle Lf, g\rangle = \langle f, L^*g\rangle$ is different than the adjoint defined by the requirement that $\nu_g(Lf) = \nu_{L^*g}(f)$ (adjoints are considered in Section C.6). Essentially, we end up with different notions for concepts on $L^2(E)$ depending on whether we regard $L^2(E)$ as a Hilbert space under the inner product, or a member of the class of Banach spaces $L^p(E)$ with the identification between $L^p(E)^*$ and $L^{p'}(E)$ given by U. The isomorphism $U\colon L^2(E) \to L^2(E)^*$ is different from the one given by the Riesz Representation Theorem (Exercise C.34).

A third possibility is to let the functionals on $L^p(E)$ be antilinear instead of linear. For example, we can associate $g \in L^{p'}(E)$ with the functional $\rho_g\colon L^p(E) \to \mathbb{C}$ given by

$$[f, \rho_g] \;=\; \int_E \overline{f(x)}\, g(x)\, dx.$$

Then the dual space is a space of antilinear functionals, i.e., the dual space is

$$L^p(E)^\neg \;=\; \{\rho\colon L^p(E) \to \mathbb{C} : \rho \text{ is bounded and antilinear}\}.$$

In this case, we have the following facts.

(a) $L^p(E)$, $L^{p'}(E)$, and $L^p(E)^\neg$ are linear spaces.

(b) $L^p(E)^\neg$ is the linear space whose elements are the bounded antilinear functionals on $L^p(E)$.

(c) $V\colon L^{p'}(E) \to L^p(E)^\neg$ given by $V(g) = \rho_g$ is an isometric isomorphism, and is linear.

While V is linear, we again have a disagreement between the notation $[\cdot,\cdot]$ and the inner product on $L^2(E)$.

Despite the fact that our discussion of notation has been quite lengthy, in the end the difference between these choices comes down to nothing more than convenience — each choice makes certain formulas "pretty" and others "unpleasant." As our main concern is the use of these notations in harmonic analysis, our choice is motivated by the formulas of harmonic analysis, and in particular the Parseval formula for the Fourier transform. We choose a notation that directly generalizes the inner product, and consequently obtain the simplest notational representation for generalizing the Fourier transform to distributions and measures (see Chapters 3 and 4).