# F

# Complex Analysis

In this appendix, we collect a few basic definitions and facts from complex analysis. Complex analysis is a vast and beautiful subject. Although we make only limited use of complex analysis in this volume, there is a rich interaction between harmonic analysis and complex analysis, some of which can be seen in the texts by Dym and McKean [DM72] and Young [You01]. Some basic texts on complex analysis include Conway [Con78], Marsden and Hoffman [MH87], and Stein and Shakarchi [SS03b].

## F.1 Analytic Functions

Complex analysis is concerned with functions that map the complex plane to itself. An analytic function is one that has a complex derivative. This is a very strong requirement — no matter what "path" to the origin that we take, the limit in the definition of the derivative must exist.

**Definition F.1 (Analytic Function).** Let $\Omega \subseteq \mathbb{C}$ be open. Then a function $f\colon \Omega \to \mathbb{C}$ is *analytic* or *holomorphic* on $\Omega$ if the limit

$$f'(z) \;=\; \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{F.1}$$

exists for all $z \in \Omega$. The function $f'$ is called the *derivative* or *complex derivative* of $f$. If $f\colon \mathbb{C} \to \mathbb{C}$ is analytic on $\mathbb{C}$, then we say that $f$ is *entire*.

Note that the variable $h$ in equation (F.1) is a complex variable. The meaning of the limit is that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |h| < \delta,\; z+h \in \Omega \quad \Longrightarrow \quad \left| f'(z) - \frac{f(z+h) - f(z)}{h} \right| \;<\; \varepsilon.$$

For example, any polynomial $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ is entire, as is the exponential function $e^z$. The function $f(z) = 1/z$ is analytic on $\mathbb{C}\backslash\{0\}$. If $r > 0$, then we set $r^z = e^{z \ln r}$.

There are many equivalent formulations of analyticity, which we will not present. For us, the important fact is that analytic functions are very highly constrained. Some of the properties of analytic functions are laid out in the next theorem. In particular, an analytic function is uniquely determined by its values on any infinite set that has an accumulation point.

**Theorem F.2 (Properties of Analytic Functions).** *If $f$ is analytic on an open set $\Omega \subseteq \mathbb{C}$, then the following statements hold.*

(a) *$f'$ is analytic on $\Omega$.*

(b) *$f$ has infinitely many complex derivatives on $\Omega$.*

(c) *Let $S$ be any infinite subset of $\Omega$ that has an accumulation point in $\Omega$. If $g$ is also analytic on $\Omega$ and $f(z) = g(z)$ for all $z \in S$, then $f(z) = g(z)$ for all $z \in \Omega$.*

We also have the following two important properties of analytic functions.

**Theorem F.3 (Liouville's Theorem).** *If $f$ is both bounded and analytic on all of $\mathbb{C}$, then $f$ is constant.*

**Theorem F.4 (Maximum Modulus Principle).** *Let $\Omega$ be a bounded, open, and connected subset of $\mathbb{C}$. Suppose that $f$ is analytic on $\Omega$ and continuous on $\overline{\Omega}$, and let $M = \sup_{z \in \partial\Omega} |f(z)|$ be the maximum of $|f|$ on the boundary of $\Omega$. Then $|f(z)| \le M$ for all $z \in \Omega$. Moreover, if $|f(z)| = M$ for some $z \in \Omega$ then $f$ is constant.*

## F.2 Power Series and Taylor Series

There are many useful connections between analytic functions and infinite series. First, we can use power series to construct analytic functions. A power series is any formal series of the form

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n z^n, \tag{F.2}$$

where $a_n \in \mathbb{C}$ are fixed.

We will let $D_r(z) = \{w \in \mathbb{C} : |w - z| < r\}$ denote the open disk in the complex plane of radius $r$ centered at $z$.

**Theorem F.5.** *Let $a_n \in \mathbb{C}$ be given for $n \ge 0$. Then there exists an $R \in [0, \infty]$ (called the* radius of convergence*) such that the following facts hold.*

(a) *The power series in equation* (F.2) *converges absolutely for all complex $z$ with $|z| < R$, and diverges whenever $|z| > R$. Furthermore, it converges uniformly on compact subsets of $D_R(0)$.*

(b) *The function $f$ defined by equation* (F.2) *is analytic on $D_R(0)$.*

(c) *The complex derivative of $f$ is*

$$f'(z) \;=\; \sum_{n=1}^{\infty} n a_n z^{n-1},$$

*and this power series has exactly the same radius of convergence $R$.*

Clearly, by replacing $z$ by $z - z_0$, we can create power series that converge in a disk centered at $z_0$.

Conversely, every analytic function can be represented as a power series. Indeed, analyticity can be defined in terms of the existence of convergent power series expansions.

**Theorem F.6 (Taylor Series).** *Suppose $\Omega \subseteq \mathbb{C}$ is open, and $f\colon \Omega \to \mathbb{C}$ is analytic. If $D_r(z_0) \subseteq \Omega$, then*

$$f(z) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z-z_0)^n, \qquad z \in D_r(z_0). \tag{F.3}$$

*This series converges absolutely on $D_r(z_0)$, and uniformly on any compact subset of $D_r(z_0)$.*

The series in equation (F.3) is called the *Taylor series* representation of $f$ about the point $z_0$.

*Example F.7.* The exponential function is

$$e^z \;=\; \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbb{C}.$$

This series converges on the entire complex plane, and $e^z$ is analytic on $\mathbb{C}$.

## F.3 Dirichlet Series

Another important type of series is a *Dirichlet series*, which is a formal series of the form

$$f(z) \;=\; \sum_{n=1}^{\infty} a_n n^z, \tag{F.4}$$

where $a_n \in \mathbb{C}$ are fixed.

Note that $n^z = e^{z \ln n}$, which is analytic on $\mathbb{C}$. Also, $|n^z| = |n^{\mathrm{Re}(z)}|$, so if $\sup |a_n| < \infty$, then the Dirichlet series in equation (F.4) converges uniformly on compact subsets of the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$, and therefore defines an analytic function $f(z)$ on this half-plane. The following result gives us a more subtle fact about Dirichlet series.

**Theorem F.8.** *If* $a_n \in \mathbb{C}$ *satisfy*

$$\sup_{n>0,\, k\geq 0} |a_n + \cdots + a_{n+k}| \;<\; \infty,$$

*then the Dirichlet series in equation* (F.4) *converges and is analytic on the half-plane* $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

The most famous Dirichlet series defines the *Riemann zeta function*, which is

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}(s) > 1. \tag{F.5}$$

It is traditional to use the letter $s$ as the variable when dealing with this function. The Riemann zeta function can be extended to a function that is analytic on all of $\mathbb{C}\backslash\{1\}$, with a simple pole at $s = 1$. This extended function is still called the Riemann zeta function, but the series definition in equation (F.5) is only valid for $\mathrm{Re}(s) > 1$ (compare Problem 1.6).

Aside from "trivial zeros" at $s = -2, -4, \dots$, it is known that any other zeros of $\zeta$ are restricted to the "critical strip" $\{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1\}$, and furthermore they are symmetrically distributed about the "critical line" $\{s \in \mathbb{C} : \mathrm{Re}(s) = 1/2\}$. The *Riemann hypothesis*, formulated by Bernhard Riemann in 1859, states that all nontrivial zeros of $\zeta(s)$ lie on the critical line. It is still unknown whether this conjecture is true or false.

## F.4 Trigonometric Polynomials

**Definition F.9.** Given $\xi \in \mathbb{R}$, set $e_\xi(x) = e^{2\pi i \xi x}$. A *trigonometric polynomial* is any finite linear combination of the functions $e_\xi$, i.e., it has the form

$$p(x) \;=\; \sum_{k=1}^{N} c_k \, e^{2\pi i \xi_k x}, \qquad x \in \mathbb{R}, \tag{F.6}$$

for some $c_k \in C$ and $\xi_k \in \mathbb{R}$.

Some authors restrict the terminology "trigonometric polynomial" to the case where the frequencies $\xi_k$ are all integers, and refer to a function of the form given in equation (F.6) with arbitrary $\xi_k$ as a *nonharmonic trigonometric polynomial*.

If we extend the domain of $e_\xi$ from $\mathbb{R}$ to $\mathbb{C}$, the resulting function

$$e_\xi(z) \;=\; e^{2\pi i \xi z}, \qquad z \in \mathbb{C},$$

is entire. Consequently, any trigonometric polynomial $p$ has an analytic extension to $\mathbb{C}$. Hence, if $p$ is not identically zero, then its zeros cannot have an accumulation point. In particular, $p$ can only have countably many zeros, and so $p$ is zero only on a set of measure zero. This gives us the following simple but useful consequence.

**Corollary F.10.** *Let $E \subseteq \mathbb{R}$ be any Lebesgue measurable set with $|E| > 0$. Then the collection $\{e_\xi\}_{\xi \in \mathbb{R}}$, where $e_\xi(x) = e^{2\pi i \xi x}$ for $x \in \mathbb{R}$, is finitely linearly independent in $L^p(E)$ for each $1 \le p \le \infty$.*

## F.5 Interpolation

In this section we will prove the Riesz–Thorin Interpolation Theorem, which is used in Chapter 1 to prove that $\{e^{2\pi i n x}\}_{n \in \mathbb{Z}}$ is a Schuader basis for $L^p(\mathbb{T})$ for $1 < p < \infty$, and in Chapter 2 to prove the Hausdorff–Young Theorem. The statement of the Riesz–Thorin Theorem is concerned with boundedness of operators on $L^p$ spaces and as such may not appear to explicitly involve complex analysis. However, the standard proofs are based on properties of analytic functions. Indeed, the Riesz–Thorin Theorem is the prototypical example of a *complex interpolation theorem* dealing with boundedness of operators on complex Banach spaces. For details on real and complex interpolation methods, we refer to the text by Bergh and Löfström [BeL76]. The proof of the Riesz–Thorin Interpolation Theorem that we give is based on Folland's presentation in [Fol99].

We first need the following result about functions analytic in a strip in the complex plane.

**Theorem F.11 (Three Lines Lemma).** *Let $f$ be a function that is bounded and continuous on the closed strip $\Omega = \{z \in \mathbb{C} : 0 \le \operatorname{Re}(z) \le 1\}$ and analytic on the interior of this strip. Set*

$$M_0 \;=\; \sup_{\operatorname{Re}(z)=0} |f(z)| \qquad \text{and} \qquad M_1 \;=\; \sup_{\operatorname{Re}(z)=1} |f(z)|.$$

*Then*

$$|f(z)| \le M_0^{1-\theta} M_1^{\theta} \quad \text{for } \operatorname{Re}(z) = \theta,\ 0 < \theta < 1.$$

*Proof. Step 1.* Assume first that $M_0,\, M_1 \le 1$ and that

$$\lim_{b \to \pm\infty} \Bigl(\sup_{0 \le a \le 1} |f(a+ib)|\Bigr) \;=\; 0.$$

Choose $R$ large enough that $|f(z)| \le 1$ for all $z \in \Omega$ with $\operatorname{Im}(z) = \pm R$. Then by the Maximum Modulus Principle (Theorem F.4) we have $|f(z)| \le 1$ for all $z$ in the rectangle $\{z = a + ib : 0 \le a \le 1,\ -R \le b \le R\}$. Letting $R \to \infty$ we obtain $|f(z)| \le 1$ for all $z \in \Omega$, which establishes the result for this $f$.

*Step 2.* Now let $f$ be an arbitrary function satisfying the hypotheses of the theorem. Given $\varepsilon > 0$, set

$$f_\varepsilon(z) \;=\; M_0^{z-1}\, M_1^{-z}\, e^{\varepsilon z (z-1)}\, f(z).$$

Exercise: Show that $f_\varepsilon$ satisfies the extra conditions required in Step 1.

Therefore $|f_\varepsilon(z)| \le 1$ everywhere on $\Omega$. Hence if $\mathrm{Re}(z) = \theta$ then

$$|f(z)|\, M_0^{\theta-1}\, M_1^{\theta} \;=\; \lim_{\varepsilon\to 0} |f_\varepsilon(z)| \;\le\; 1. \qquad \square$$

**Theorem F.12 (Riesz–Thorin Interpolation Theorem).** *Let* $1 \le p_0$, $p_1$, $q_0$, $q_1 \le \infty$ *be given. Assume that* $T\colon L^{p_0}(\mathbb{R}) + L^{p_1}(\mathbb{R}) \to L^{q_0}(\mathbb{R}) + L^{q_1}(\mathbb{R})$ *is linear with* $T(L^{p_0}(\mathbb{R})) \subseteq L^{q_0}(\mathbb{R})$ *and* $T(L^{p_1}(\mathbb{R})) \subseteq L^{q_1}(\mathbb{R})$, *and assume that the operator norms*

$$M_0 \;=\; \|T\|_{L^{p_0}\to L^{q_0}} \qquad \text{and} \qquad M_1 \;=\; \|T\|_{L^{p_1}\to L^{q_1}}$$

*are finite. Given any* $0 < \theta < 1$, *if we define* $p$, $q$ *by*

$$\frac{1}{p} \;=\; \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} \;=\; \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \tag{F.7}$$

*then* $T\colon L^p(\mathbb{R}) \to L^q(\mathbb{R})$ *is bounded and satisfies*

$$\|T\|_{L^p\to L^q} \;\le\; M_0^{1-\theta}\, M_1^{\theta}.$$

*Proof.* Let $0 < \theta < 1$ be fixed, and let $p$, $q$ be as defined in equation (F.7). Note that $L^p(\mathbb{R}) \subseteq L^{p_0}(\mathbb{R}) + L^{p_1}(\mathbb{R})$ by Problem B.15, so $T$ is well-defined on $L^p(\mathbb{R})$.

*Case 1*: $p_0 = p_1$. In this case $p = p_0 = p_1$, so it follows from Problem B.14 that

$$\|Tf\|_q \;\le\; \|Tf\|_{q_0}^{1-\theta}\, \|Tf\|_{q_1}^{\theta} \;\le\; M_0^{1-\theta}\, M_1^{\theta}\, \|f\|_p.$$

Thus $T$ maps $L^p(\mathbb{R})$ boundedly into $L^q(\mathbb{R})$.

*Case 2*: $p_0 \ne p_1$ and $1 < q \le \infty$. It follows from equation (F.7) that $p < \infty$ in this case, and therefore the set $S$ of all simple functions that have compact support is dense in $L^p(\mathbb{R})$ by Exercise B.59. Appealing to [Fol99, Thm. 6.14],

$$\|Tf\|_q \;=\; \sup\biggl\{\biggl|\int Tf(x)\, g(x)\, dx\biggr| : g \in S,\ \|g\|_{q'} = 1\biggr\}. \tag{F.8}$$

Fix any $f \in S$ with $\|f\|_p = 1$, and suppose that $g \in S$ satisfies $\|g\|_{q'} = 1$. Since $f$ and $g$ are both simple functions, we can write them as

$$f \;=\; \sum_{j=1}^{M} c_j \chi_{E_j} \qquad \text{and} \qquad g \;=\; \sum_{k=1}^{N} d_k \chi_{F_k}.$$

We can assume that the scalars $c_j$, $d_k$ are all nonzero, the sets $E_j$ are disjoint, and the sets $F_k$ are disjoint. Write $c_j = a_j\,|c_j|$ and $d_k = b_k\,|d_k|$ where $a_j$ and $b_k$ are complex numbers with unit modulus. Set

$$\alpha(z) \;=\; \frac{1-z}{p_0} + \frac{z}{p_1} \qquad \text{and} \qquad \beta(z) \;=\; \frac{1-z}{q_0} + \frac{z}{q_1},$$

and note that $\alpha(\theta) = 1/p$ and $\beta(\theta) = 1/q$. In particular, $0 < \alpha(\theta), \beta(\theta) \le 1$. Define

$$f_z \;=\; \sum_{j=1}^{M} a_j\,|c_j|^{\alpha(z)/\alpha(\theta)}\,\chi_{E_j} \quad\text{and}\quad g_z \;=\; \sum_{k=1}^{N} b_k\,|d_k|^{(1-\beta(z))/(1-\beta(\theta))}\,\chi_{F_k},$$

and set

$$\phi(z) \;=\; \int Tf_z(x)\,g(x)\,dx \;=\; \sum_{j=1}^{M}\sum_{k=1}^{N} C_{jk}\,|c_j|^{\alpha(z)/\alpha(\theta)}\,|d_k|^{(1-\beta(z))/(1-\beta(\theta))},$$

where

$$C_{jk} \;=\; a_j b_k \int T\chi_{E_j}(x)\,\chi_{F_k}(x)\,dx.$$

Since $\alpha$ and $\beta$ are polynomials in $z$, the function $\phi$ is entire. Exercise: Show that $\phi$ is bounded on the closed strip

$$\Omega \;=\; \{z \in \mathbb{C} : 0 \le \operatorname{Re}(z) \le 1\}.$$

Suppose that $\operatorname{Re}(z) = 0$, i.e., $z = ib$ where $b \in \mathbb{R}$. Note that

$$\alpha(ib) \;=\; \frac{1-ib}{p_0} + \frac{ib}{p_1} \;=\; \frac{1}{p_0} + ib\left(\frac{1}{p_1} - \frac{1}{p_0}\right).$$

Assume for the moment that $p_0, q_0' < \infty$. Then since the sets $E_j$ are disjoint, we have that

$$|f_{ib}| \;=\; \sum_{j=1}^{M} \big||c_j|^{\alpha(ib)/\alpha(\theta)}\big|\,\chi_{E_j} \;=\; \sum_{j=1}^{M} |c_j|^{p/p_0}\,\chi_{E_j} \;=\; |f|^{p/p_0},$$

and a similar calculation shows that $|g_{ib}| = |g|^{q'/q_0'}$. Therefore

$$\begin{aligned}
|\phi(z)| \;&=\; \left|\int Tf_{ib}(x)\,g_{ib}(x)\,dx\right| \\
&\le\; \|Tf_{ib}\|_{q_0}\,\|g_{ib}\|_{q_0'} \\
&\le\; M_0\,\|f_{ib}\|_{p_0}\,\|g_{ib}\|_{q_0'} \\
&=\; M_0\,\big\|\,|f|^{p/p_0}\,\big\|_{p_0}\,\big\|\,|g|^{q'/q_0'}\,\big\|_{q_0'} \\
&=\; M_0\,\|f\|_p^{p/p_0}\,\|g\|_{q'}^{q'/q_0'} \\
&=\; M_0.
\end{aligned}$$

A small modification of this argument shows that the conclusion $|\phi(z)| \le M_0$ for $\operatorname{Re}(z) = 0$ still holds when $p_0 = \infty$ or $q_0' = \infty$, and a similar calculation shows that $|\phi(z)| \le M_1$ for $\operatorname{Re}(z) = 1$ (exercises).

The Three Lines Lemma therefore implies that $|\phi(\theta)| \le M_0^{1-\theta}\, M_1^{\theta}$. However, $f_\theta = f$ and $g_\theta = g$ so we have

$$\left|\int Tf(x)\, g(x)\, dx\right| \;=\; |\phi(\theta)| \;\le\; M_0^{\theta}\, M_1^{1-\theta}.$$

Applying equation (F.8), we conclude that $\|Tf\|_q \le M_0^{1-\theta}\, M_1^{\theta}$ for all unit vectors $f \in S$.

Since $S$ is dense in $L^p(\mathbb{R})$, we know that $T$ has a unique extension to a bounded operator on $L^p(\mathbb{R})$, but the problem is that we do know not know if this extension agrees with the operator $T$ given in the hypotheses of the theorem. So, we still must verify that the *original* operator $T$ is in fact bounded on all of $L^p(\mathbb{R})$. To address this point, fix $f \in L^p(\mathbb{R})$. Then we can find simple functions $f_n \in S$ such that $f_n \to f$ pointwise a.e. with $|f_n| \le |f|$ a.e. for every $n$, and furthermore the convergence is uniform on any set where $f$ is bounded (see Problem B.7). Since $p$ is finite, the Lebesgue Dominated Convergence Theorem therefore implies that $f_n \to f$ in $L^p$-norm.

Set $E = \{x \in \mathbb{R} : |f(x)| > 1\}$, and define

$$g = f\,\chi_E, \qquad g_n = f_n\,\chi_E, \qquad h = f - g, \qquad h_n = f_n - g_n.$$

At least one of $p_0$, $p_1$ is finite, so without loss of generality let us say that it is $p_0$. Since $f$ is bounded on the set $\mathbb{R}\backslash E$, which has finite measure, we have

$$\|Th - Th_n\|_{q_1} \;\le\; M_0\, \|h - h_n\|_{p_1} \;\le\; M_0\, |\mathbb{R}\backslash E|^{1/p_1}\, \|h - h_n\|_\infty \;\to\; 0 \;\text{ as } n \to \infty.$$

Additionally, by the Lebesgue Dominated Convergence Theorem,

$$\|Tg - Tg_n\|_{q_0} \;\le\; M_0\, \|g - g_n\|_{p_0} \;\to\; 0 \quad \text{as } n \to \infty.$$

Since $L^p$-norm convergence implies pointwise a.e. convergence of a subsequence, by replacing $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ with appropriate subsequences we may assume that $Th_n \to h$ pointwise a.e. and $Tg_n \to g$ pointwise a.e. Since $T$ is linear, this implies that $Tf_n \to f$ pointwise a.e. Applying Fatou's Lemma, we conclude that

$$\begin{aligned}
\|Tf\|_q \;=\; \left(\int |Tf(x)|^q\, dx\right)^{1/q} \;&\le\; \liminf_{n\to\infty} \left(\int |Tf_n(x)|^q\, dx\right)^{1/q} \\
&=\; \liminf_{n\to\infty}\, \|Tf_n\|_q \\
&=\; \liminf_{n\to\infty}\, M_0^{1-\theta}\, M_1^{\theta}\, \|f_n\|_p \\
&=\; M_0^{1-\theta}\, M_1^{\theta}\, \|f\|_p.
\end{aligned}$$

Consequently $T$ is a bounded mapping of $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$, and $\|T\|_{L^p\to L^q} \le M_0^{1-\theta}\, M_1^{\theta}$.

*Case 3*: $p_0 \ne p_1$ and $q = 1$. The proof for Case 2 can be adapted by setting $g_z = g$ for all $z$ (exercise). □