FUNCTIONAL ANALYSIS LECTURE NOTES:

PROBLEMS ON c AND c_0

CHRISTOPHER HEIL

Definition 1. Sequences with unspecified limits are indexed by the natural numbers $\mathbb{N}$. We set

$$\begin{aligned} c \;&=\; c(\mathbb{N}) \;=\; \Big\{a = (a_k) : \lim_{k\to\infty} a_k \text{ exists}\Big\}, \\ c_0 \;&=\; c_0(\mathbb{N}) \;=\; \Big\{a = (a_k) : \lim_{k\to\infty} a_k = 0\Big\}, \\ c_{00} \;&=\; c_{00}(\mathbb{N}) \;=\; \Big\{a = (a_k) : \text{only finitely many } a_k \text{ are nonzero}\Big\}. \end{aligned}$$

Definition 2 (Basis)**.** A countable sequence $\{x_n\}$ in a Banach space X is a *basis* for X if

$$\forall\, x \in X, \quad \exists \text{ unique scalars } a_n(x) \text{ such that } x \;=\; \sum_n a_n(x)\, x_n. \tag{1}$$

We call the series in equation (1) the *basis expansion* or *basis representation* of x with respect to $\{x_n\}$.

Definition 3. Let $\{x_n\}$ be a basis for a Banach space X.

(a) $\{x_n\}$ is an *unconditional basis* if the series in equation (1) converge unconditionally for each $x \in X$, i.e., if $\sigma \colon \mathbb{N} \to \mathbb{N}$ is any bijection then the series $\sum a_\sigma(n)(x)\, x_\sigma(n)$ converges. A basis that is not an unconditional basis is called a *conditional basis*.

(b) $\{x_n\}$ is an *absolutely convergent basis* if the series in equation (1) converge absolutely for each $x \in X$.

(c) $\{x_n\}$ is a *bounded basis* if $\{x_n\}$ is norm-bounded both above and below, i.e., if $0 < \inf \|x_n\| \le \sup \|x_n\| < \infty$.

(d) $\{x_n\}$ is a *normalized basis* if $\{x_n\}$ is normalized, i.e, if $\|x_n\| = 1$ for every $n \in \mathbb{N}$.

Absolutely convergent bases have a simple characterization: A Banach space X has an absolutely convergent basis if and only if X is topologically isomorphic to ℓ^1 (exercise).

Problem 1. (a) Show that c and c_0 are closed subspaces of ℓ^∞.

(b) Show that c_{00} is a proper, dense subspace of c_0, and hence is not closed with respect to $\|\cdot\|_{\ell^\infty}$.

(c) Let $\{\delta_n\}$ denote the sequence of standard basis vectors. Given $x = (x_n) \in c_0$, show that $x = \sum x_n \delta_n$, where the series converges with respect to the norm $\|\cdot\|_{\ell^\infty}$. Show further that the scalars x_n in this representation are unique.

Problem 2. Given $y = (y_k) \in \ell^1$, define $\mu_y \colon c_0 \to \mathbb{F}$ by $\langle x, \mu_y \rangle = \sum x_k y_k$. Show that $y \mapsto \mu_y$ is an isometric isomorphism of ℓ^1 onto ${c_0}^*$. Thus ${c_0}^* \cong \ell^1$.

Problem 3. In this problem, we will denote the components of $x \in \ell^p$ by $x = \big(x(k)\big)$.

(a) Given $1 < p < \infty$ and $x_n,\, y \in \ell^p$, show that $x_n \xrightarrow{\text{w}} y$ if and only if $\sup \|x_n\|_{\ell^p} < \infty$ and x_n *converges componentwise* to y, i.e., $\lim_{n\to\infty} x_n(k) = y(k)$ for each $k \in \mathbb{N}$. Do both implications remain valid if $p = 1$?

(b) Recall that $\ell^1 \cong {c_0}^*$. Given $x_n,\, y \in \ell^1$, show that $x_n \xrightarrow{\text{w*}} y$ if and only if x_n converges componentwise to y and $\sup \|x_n\|_{\ell^1} < \infty$.

Problem 4. (a) Show that the standard basis $\{\delta_n\}_{n\in\mathbb{N}}$ is a normalized unconditional basis for ℓ^p for each $1 \le p < \infty$, and is also a normalized unconditional basis for c_0.

(b) By Problem 1, $c = \{x = (x_n) \in \ell^\infty : \lim_{n\to\infty} x_n \text{ exists}\}$ is a closed subspace of ℓ^∞. Find a vector $\delta_0 \in c$ such that $\{\delta_n\}_{n\ge 0}$ is a normalized unconditional basis for c.

(c) Show that c^* is isometrically isomorphic to ℓ^1. Compare Problem 2, which shows that we also have ${c_0}^* \cong \ell^1$, and Problem 7, which shows that c and c_0 are topologically isomorphic but not isometrically isomorphic.

Problem 5. For each $n \in \mathbb{N}$, define $y_n = (1, \dots, 1, 0, 0, \dots)$, where the 1 is repeated n times. Show that $\{y_n\}$ is a normalized conditional basis for c_0.

Problem 6. For each $n \in \mathbb{N}$, define $z_n = (0, \dots, 0, 1, 1, \dots)$, where the 0 is repeated $n - 1$ times. Show that $\{z_n\}$ is a normalized conditional basis for c (this is called the *summing basis* for c).

Problem 7. (a) Let $\{\delta_n\}$ be the standard basis for c_0. By Problem 4, if we set $\delta_0 = (1, 1, \dots)$, then $\{\delta_n\}_{n\ge 0}$ is a basis for c. Show that c and c_0 are topologically isomorphic, and that these two bases are equivalent.

(b) Show that if $x \in c_0$ and $\|x\|_\infty = 1$, then there exist $y \ne z \in c_0$ with $\|y\|_\infty = \|z\|_\infty = 1$ such that $x = (y + z)/2$. Show that the analogous statement for c fails (thus c is *strictly convex* while c_0 is not).

(c) Show that c is not isometrically isomorphic to c_0 (even so, note that their dual spaces c^* and ${c_0}^*$ are each isometrically isomorphic to ℓ^1 by Problem 2 and 4).