Finite-Dimensional TVS

We know that all norms on $\mathbb{F}^n$ are equivalent and every finite-dimensional normed space is topologically isomorphic to $\mathbb{F}^n$ for some n.
We now investigate finite-dimensional TVS.

Lemma
If X is a TVS then every linear map $L: \mathbb{F}^n \to X$ is continuous.

Proof:
Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{F}^n$.

Given $x \in \mathbb{F}^n$, write $x = (x_1, \dots, x_n)$.

Fix $1 \le k \le n$. Then $\pi_k(x) = x_k$ is a linear map of the normed space $\mathbb{F}^n$ to the normed space $\mathbb{F}$.

As $\mathbb{F}^n$ is finite-dimensional, π_k is continuous.

As scalar multiplication & vector addition are continuous on X, it follows that

$$x \mapsto L(x) = x_1 L(e_1) + \cdots + x_n L(e_n)$$

is continuous on $\mathbb{F}^n$. ∎

Theorem

Let X be a TVS. If Y is an n-dimensional subspace of X then:

(a) Y is closed,

(b) Every linear bijection $L: \mathbb{F}^n \to Y$ is a topological isomorphism.

Proof:

(b) Let
$$S = \{ x \in \mathbb{F}^n : \|x\| = 1 \},$$

where $\|\cdot\|$ is the Euclidean (ℓ^2) norm on $\mathbb{F}^n$.

Suppose $L: \mathbb{F}^n \to Y$ is a linear bijection. By the lemma, L is continuous, so our goal is to show that L^{-1} is continuous. Since S is compact & L is continuous, $K = L(S)$ is compact in Y. Since L is injective & $0 \notin S$, we have $0 \notin K$. Therefore $X \setminus K$ is an open neighborhood of 0, so $\exists$ balanced open neighborhood V s.t. $0 \in V \subseteq X \setminus K$.

Consider $E = L^{-1}(V)$. Since L maps into Y,

$$E = L^{-1}(V) = L^{-1}(V \cap Y).$$

If $x \in S$ then $Lx \in K$, which is disjoint from V.

Hence S is disjoint from E. Now, V and Y are balanced and L^{-1} is linear, so $E = L^{-1}(V \cap Y)$ is a balanced subset of $\mathbb{F}^n$, and therefore is connected. Since $0 \in E$, E is connected, and E is disjoint from the unit sphere S, we must have

$$E \subseteq B = \{x \in \mathbb{F}^n : \|x\| < 1\}.$$

Thus $L^{-1} : Y \to \mathbb{C}^n$ maps $V \cap Y$, which is an open neighborhood of 0 in Y, into the bounded set E. A previous theorem therefore implies that L^{-1} is continuous.

(a) Choose any $y \in \overline{Y}$. Since Y is n-dimensional, $\exists$ linear bijection $L : \mathbb{F}^n \to Y$. Let V be as in the proof of part (b). By an earlier result $\bigcup_{n=1}^{\infty} nV = X$, so $\exists\, t > 0$ s.t. $y \in tV$. Note that $t\overline{B}$ is compact, so since $L : \mathbb{F}^n \to X$ is continuous, $L(t\overline{B})$ must be a compact subset of X, and hence is closed. Therefore

$$
\begin{aligned}
y &\in \overline{Y} \cap tV \\
&\subseteq \overline{Y \cap tV} \qquad \leftarrow \text{Rudin claims} \\
&= \overline{tY \cap tV} \\
&= \overline{L(tE)} \\
&\subseteq \overline{L(tB)} \\
&\subseteq \overline{L(t\overline{B})} \\
&= L(t\overline{B}) \\
&\subseteq Y.
\end{aligned}
$$

Hence Y is closed. ∎

Lemma

Let X be a TVS and fix $A \subseteq X$. Then

$$\bar{A} = \bigcap \{A + V : V \text{ is an open neighborhood of } 0\}$$

Proof:

$\supseteq$ Suppose $x \in A + V$ for every open neighborhood V of 0. If $x \in A$ then we have $x \in \bar{A}$, so suppose that $x \notin A$. Let U be any open neighborhood of x. Then $x - U$ is an open neighborhood of 0, so we must have $x \in A + (x - U)$. This implies $\exists a \in A$, $u \in U$ such that $x = a + x - u$, so $u = a \in A$. We must have $u \neq x$ since $x \notin A$, so every open neighborhood of x contains a point of A other than x itself. Hence x is an accumulation point of A, so $x \in \bar{A}$.

$\subseteq$ Exercise. $\blacksquare$

<u>Theorem</u>

If X is a TVS, then

X is locally compact $\Longleftrightarrow$ X is finite-dimensional.

<u>Proof:</u>

$\Leftarrow$ If $\dim(X) = n$ then $\exists$ linear bijection $L : \mathbb{F}^n \to X$.

By a previous theorem L must be a topological isomorphism.

Since $\mathbb{F}^n$ is locally compact, X must be also.

$\Rightarrow$ Suppose X is locally compact. Then by definition

$\exists$ open neighborhood V of 0 such that $\overline{V}$ is compact.

Earlier theorems then imply that V is bounded and

$\{2^{-n}V\}_{n \in \mathbb{N}}$ is a local base for X. Since

$$\overline{V} \subseteq X = \bigcup_{x \in X} \left(\tfrac{1}{2}V + x\right),$$

$\exists\, x_1, \ldots, x_n \in X$ such that

$$\overline{V} \subseteq \bigcup_{k=1}^{n} \left(\tfrac{1}{2}V + x_k\right). \qquad (*)$$

Set $Y = \operatorname{span}\{x_1, \ldots, x_n\}$, so $\dim(Y) \le n$.

By an earlier theorem, every finite-dimensional subspace of a TVS is closed, so Y is closed.

By equation (*), we have

$$V \subseteq \overline{V} \subseteq \bigcup_{k=1}^{n} \left(\tfrac{1}{2}V + x_k\right) \subseteq \tfrac{1}{2}V + Y.$$

Iterating,

$$V \subseteq \tfrac{1}{2}V + Y \subseteq \tfrac{1}{2}\left(\tfrac{1}{2}V + Y\right) + Y = \tfrac{1}{4}V + Y,$$

etc. Taking into account the preceding lemma and the fact that $\{2^{-n}V\}_{n \in \mathbb{N}}$ is a local base, it follows that

$$V \subseteq \bigcap_{n=1}^{\infty} \left(2^{-n}V + Y\right) = \overline{Y} = Y.$$

Thus the subspace Y contains an open neighborhood of 0. But then

$$Y \supseteq \bigcup_{k=1}^{\infty} kV = X,$$

so $X = Y$ is finite-dimensional. $\blacksquare$