MATH 7338 HOMEWORK #1 Due date: September 22, 2009

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let $\mathcal{E}$ be a collection of subsets of a set X whose union is X. The topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ is defined to be the smallest topology on X that contains $\mathcal{E}$ (which means that $\mathcal{T}(\mathcal{E})$ is the intersection of all the topologies $\mathcal{T}$ that contain $\mathcal{E}$ as a subset).

Show that the topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ is set of all unions of finite intersections of elements of $\mathcal{E}$:

$$\mathcal{T}(\mathcal{E}) = \left\{ \bigcup_{i \in I} \bigcap_{j=1}^{n} E_{ij} : I \text{ arbitrary}, \ n \in \mathbb{N}, \ E_{ij} \in \mathcal{E} \right\}.$$

2. Let X and Y be topological spaces, and set

$$\mathcal{B} = \{U \times V : \text{open } U \subseteq X, \text{ open } V \subseteq Y\}.$$

We define the *product topology* on $X \times Y$ to be the topology $\mathcal{T}(\mathcal{B})$ generated by $\mathcal{B}$.

(a) Show that $\mathcal{B}$ as given above is a *base* for $\mathcal{T}(\mathcal{B})$, which means that if $W \in \mathcal{T}(\mathcal{B})$, then there exist sets U_α open in X and V_α open in Y such that $W = \cup_\alpha (U_\alpha \times V_\alpha)$.

(b) Show that if $W \subseteq X \times Y$ is open with respect to the product topology, then for each $x \in X$ the restriction $W_x = \{y \in Y : (x, y) \in W\}$ is open in Y, and likewise for each $y \in Y$ the restriction $W^y = \{x \in X : (x, y) \in W\}$ is open in X.

3. Let $\{x_i\}_{i \in I}$ be a Hamel basis for an infinite-dimensional Banach space X, which means that each $x \in X$ can be written uniquely as $x = \sum_{i \in I} a_i(x)\, x_i$ with at most finitely many of the scalars $a_i(x)$ are nonzero. Each a_i is a linear functional on X, and we call $\{a_i\}_{i \in I}$ the associated sequence of coefficient functionals.

(a) Show by example that it is possible for some particular functional a_i to be continuous.
Hint: Try $X = \ell^1$.

(b) Show that $a_i(x_j) = \delta_{ij}$ for $i, j \in I$, where δ_{ij} is the Kronecker delta.

(c) Show that at most finitely many functionals a_i can be continuous, i.e., J is finite.

(d) Let $J = \{i \in I : a_i \text{ is continuous}\}$. Show that $\sup_{j \in J} \|a_j\| < \infty$.
Hint: Uniform Boundedness Principle.

(e) Give an example of an infinite-dimensional normed linear space that has a Hamel basis $\{x_i\}_{i \in I}$ such that each of the associated coefficient functionals a_i for $i \in I$ is continuous.

4. Let X be an infinite-dimensional Banach space. We showed in class that there exists a linear functional $\mu\colon X \to \mathbb{C}$ that is unbounded. The Cartesian product $X_1 = X \times \mathbb{C}$ is a Banach space with respect to the norm $\|(x,c)\|_{X_1} = \|x\|_X + |c|$ (you can take this as given). Set $Y = \text{graph}(\mu) = \{(x,\mu(x)) : x \in X\}$, and define $\|(x,\mu(x))\|_Y = \|x\|_X$.

(a) Show that $(Y, \|\cdot\|_Y)$ is a Banach space.

(b) Show that even though $Y \subseteq X_1$, the normed space $(Y, \|\cdot\|_Y)$ is not continuously embedded into $(X_1, \|\cdot\|_{X_1})$, i.e., the mapping $I\colon (Y, \|\cdot\|_Y) \to (X_1, \|\cdot\|_{X_1})$ given by $I(z) = z$ is not continuous.

Remark: This is interesting because in many "practical" circumstances, you can use the Closed Graph Theorem to show that if one Banach space is contained in another then the embedding is continuous.