MATH 7338 HOMEWORK #2 Due date: October 8, 2009

Work the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. (a) Let $\delta_n = (0, 0, \dots, 0, 1, 0, 0, \dots)$ and $\delta_0 = (1, 1, 1, \dots)$. Show that $\{\delta_n\}_{n\geq 0}$ is a Schauder basis for c.

Note: You can use the fact (shown in class) that $\{\delta_n\}_{n\geq 1}$ is a Schauder basis for c_0, so this part should be easy.

(b) Show that c^* is isometrically isomorphic to ℓ^1.

(c) Show that c and c_0 are topologically isomorphic and further that the two bases described above are *equivalent* in the sense that there exists a topological isomorphism $T\colon c \to c_0$ that maps the basis $\{\delta_n\}_{n\geq 0}$ for c onto the basis $\{\delta_n\}_{n\geq 1}$ for c_0.

(d) Show that if $x \in c_0$ and $\|x\|_\infty = 1$, then there exist $y \neq z \in c_0$ with $\|y\|_\infty = \|z\|_\infty = 1$ such that $x = (y+z)/2$. Show that the analogous statement for c can fail.

(e) Show that c is not isometrically isomorphic to c_0 (even so, their dual spaces c^* and ${c_0}^*$ are each isometrically isomorphic to ℓ^1!).

2. Recall that the *total variation* $|\nu|$ of a complex Borel measure ν is the positive measure $d|\nu| = |f|\,d\mu$, where μ is any positive measure and f is any function in $L^1(\mu)$ such that $d\nu = f\,d\mu$. (You can assume that such μ, f exist and that $|\nu|$ is independent of the choice of μ and f.) Prove the following statements hold.

(a) $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{B}_\sigma$.

(b) $\nu \ll |\nu|$, and there exists g with $|g| = 1$ $|\nu|$-a.e. such that $d\nu = g\,d|\nu|$.

Hint: Apply Radon–Nikodym, and show that $\int_E (|g|-1)\,d|\nu| = 0$ for every Borel set E. This implies (why?) that $|g| = 1$ $|\nu|$-a.e.

(c) If $f \in L^1(\nu)$, then $\left|\int f\,d\nu\right| \leq \int |f|\,d|\nu|$.

(d) $|\nu| = \mu_1 = \mu_2 = \mu_3$, where

$$\mu_1(E) = \sup\left\{\sum_{k=1}^{n} |\nu(E_k)| : n \in \mathbb{N}, E_k \in \mathcal{B}_\sigma, E = \bigcup_{k=1}^{n} E_k \text{ disjointly}\right\},$$

$$\mu_2(E) = \sup\left\{\sum_{k=1}^{\infty} |\nu(E_k)| : E_k \in \mathcal{B}_\sigma, E = \bigcup_{k=1}^{\infty} E_k \text{ disjointly}\right\},$$

$$\mu_3(E) = \sup\left\{\left|\int_E f\,d\nu\right| : |f| \leq 1 \ |\nu|\text{-a.e.}\right\}.$$

(Turn over for hints.)

Hints: Show that $\mu_1 \le \mu_2 \le \mu_3 = |\nu|$ and $\mu_3 \le \mu_1$.

$\mu_2 \le \mu_3$: Suppose that $E = \bigcup_{k=1}^\infty E_k$ disjointly. Let c_k be the scalars of unit modulus such that $\bigl|\int_{E_k} d\nu\bigr| = c_k \int_{E_k} d\nu$, and set $f = \sum c_k \chi_{E_k}$.

$|\nu| \le \mu_3$: By part (b), there exists f with $|f| = 1$ ν-a.e. such that $d\nu = f\, d|\nu|$. Then $\bar{f}\, d\nu = \bar{f} f\, d|\nu| = |f|^2\, d|\nu| = d|\nu|$.

$\mu_3 \le \mu_1$: Suppose $|f| \le 1$. Since f is bounded, there exists a sequence of simple functions that converges to f uniformly. Hence, if $\varepsilon > 0$ then there exists a simple function $\phi = \sum_{j=1}^N c_j \chi_{E_j}$ such that $\sup_{x\in\mathbb{R}} |f(x) - \phi(x)| \le \varepsilon$. Consequently, $|c_j| \le 1 + \varepsilon$ since $c_j = \phi(x)$ for some x, and $\bigl|f\, d\nu\bigr| \le \bigl|\int \phi\, d\nu\bigr| + \varepsilon\, \|\nu\|$.

3. This problem will show that the locally finite positive measures on $\mathbb{N}$ (which are precisely the Radon measures on $\mathbb{N}$) are in 1-1 correspondence with the positive linear functionals on c_{00}.

(a) Give the convergence criterion corresponding to the inductive limit topology on c_{00}.

(b) Show that if ν is a positive locally finite measure on $\mathbb{N}$, then $\langle f, \nu\rangle = \sum f(k)\, \nu\{k\}$ defines a positive linear functional on c_{00} that is continuous with respect to the inductive limit topology on c_{00}.

(c) Show that if μ is a positive linear functional on c_{00} then there exists a unique sequence of nonnegative scalars $w = (w_k)_{k\in\mathbb{N}}$ such that $\langle f, \mu\rangle = \sum f(k)\, w_k$ for $f \in c_{00}$. Show there is a unique locally finite positive measure ν on $\mathbb{N}$ such that $w_k = \nu\{k\}$ for every k.