MATH 7338 HOMEWORK #3 Due date: October 29, 2009

Work FIVE of the following problems and hand in your solutions. You may work together with other people in the class, but you must each write up your solutions independently. A subset of these will be selected for grading. Write LEGIBLY on the FRONT side of the page only, and STAPLE your pages together.

1. Let $\mathcal{F}(\mathbb{R})$ be the vector space containing all functions $f\colon \mathbb{R} \to \mathbb{C}$. For each $x \in \mathbb{R}$, define a seminorm on $\mathcal{F}(\mathbb{R})$ by $\rho_x(f) = |f(x)|$. Then convergence with respect to the family of seminorms $\{\rho_x\}_{x\in\mathbb{R}}$ corresponds to pointwise convergence of functions. Show that this topology is not normable, i.e., Show that there is no norm on $\mathcal{F}(\mathbb{R})$ that defines the same convergence criterion.

Hint: Find f_n such that $c_n f_n \to 0$ pointwise for every choice of scalars c_n.

2. Let $\{\rho_\alpha\}_{\alpha\in J}$ be a family of seminorms on a vector space X. Show that the induced topology on X is Hausdorff if and only if

$$\rho_\alpha(x) = 0 \text{ for all } \alpha \in J \quad \Longleftrightarrow \quad x = 0.$$

3. Show that if $\mu \in \mathcal{E}'(\mathbb{R})$, then there exists a compact set $K \subseteq \mathbb{R}$ such that if $f \in C^\infty(\mathbb{R})$ and $\operatorname{supp}(f) \subseteq \mathbb{R}\backslash K$, then $\langle f, \mu\rangle = 0$.

4. Let X be a normed space and S a subspace of X. Prove that the following statements are equivalent.

(a) S is strongly closed (i.e., closed with respect to the norm topology).

(b) S is weakly closed (i.e., closed with respect to the weak topology).

Hint: Suppose that S is strongly closed and choose $x \notin S$. By Hahn–Banach, there exists a $\mu \in X^*$ such that $\mu|_S = 0$ and $\langle x, \mu\rangle = 1$. Show that $U = \mu^{-1}(\mathbb{C}\backslash\{0\})$ is open in the weak topology, and use this to show that $X\backslash S$ is weakly open.

5. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for a Hilbert space H.

(a) Show that $e_n \xrightarrow{\text{w}} 0$.

(b) Show that $n^{1/2}e_n$ does not converge weakly to 0.

(c) Show that 0 belongs to the weak closure of $\{n^{1/2}e_n\}_{n\in\mathbb{N}}$, i.e., 0 is an accumulation point of this set with respect to the weak topology on H.

Hint: If U is any open neighborhood of 0 in the weak topology, then there exist $r > 0$ and $y_1, \dots, y_N$ such that the base element $B = \bigcap_{j=1}^{N} B_r^{y_j}(0)$ satisfies $0 \in B \subseteq U$. If $n^{1/2}e_n$ does not belong to B for any n, apply the Plancherel Equality to $\|y_1\|^2 + \cdots + \|y_N\|^2$ to obtain a contradiction.

6. This problem will use Alaoglu's Theorem to construct an element of $(\ell^\infty)^*$ that does not belong to ℓ^1.

(a) For each $n \in \mathbb{N}$, define $\mu_n \colon \ell^\infty \to \mathbb{C}$ by $\langle x, \mu_n\rangle = \frac{1}{n}\,(x_1+\cdots+x_n)$ for $x = (x_1, x_2, \dots) \in \ell^\infty$. Show that $\mu_n \in (\ell^\infty)^*$ and $\|\mu_n\| \le 1$.

(b) Use Alaoglu's Theorem to show that there exists a $\mu \in (\ell^\infty)^*$ that is an accumulation point of $\{\mu_n\}_{n\in\mathbb{N}}$.

(c) Show that $\mu \ne \widetilde{x}$ for any $x \in \ell^1$, where $\widetilde{x}$ is the image of x under the natural embedding of ℓ^1 into $(\ell^1)^{**} = (\ell^\infty)^*$.

Hint: Let μ be a weak* accumulation point of $\{\mu_n\}_{n\in\mathbb{N}}$. Then there must exist a net contained in $\{\mu_n\}_{n\in\mathbb{N}}$ that converges weak* to μ, say $\{\mu_{n_i}\}_{i\in I}$ where I is a directed set. Note that $\{\mu_{n_i}\}_{i\in I}$ need not be a *subsequence* of $\{\mu_n\}_{n\in\mathbb{N}}$, since repetitions are allowed in a sequence and therefore I might have any cardinality.

Let $\{e_j\}_{j\in\mathbb{N}}$ denote the standard basis vectors, which all belong to ℓ^∞. The open strip

$$B_r^{e_j}(\mu) \;=\; \bigl\{\nu \in (\ell^\infty)^* : |\langle e_j, \mu - \nu\rangle| < r\bigr\}.$$

is a weak* open neighborhood of μ, so there must be infinitely many μ_{n_i} in $B_r^{e_j}(\mu)$. Use this to show that if $\mu = \widetilde{x}$ for some $x \in \ell^\infty$, then $x = 0$ and hence $\mu = 0$.

To show that $\mu \ne 0$, consider $x = (1, 1, 1, \dots)$.

7. Show that the closed unit ball $\{x \in c_0 : \|x\|_\infty \le 1\}$ in c_0 is not weakly compact.

Hint: Find a weakly open cover that has no finite subcover.